DISCRETE MATHEMATICS 21228 — HOMEWORK 3

JC

Due in class Wednesday September 17. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

Please write the name of your recitation instructor and the time and place of your recitation at the top of your homework.

(1) Consider the experiment in which a biased coin with probability p of coming up heads is tossed N times. Write down a formula for the expectation value of the number of times the coin comes up heads. Prove that this expectation value is Np (why might we have expected this answer?)

(2) Consider the experiment in which the biased coin with probability p of coming up heads is tossed repeatedly until it comes up tails. Find a formula for p_n, the probability that the coin is tossed exactly n times. For a given value of p, what is the least N such that the probability we toss the coin at most N times is at least 0.99?

(3) Consider the 3D generalisation of the notion of path: a path is a sequence $(x_0, y_0, z_0), (x_1, y_1, z_1), \ldots (x_n, y_n, z_n)$ where at each step one of the three coordinates increases by one while the other two remain constant. If a, b, c are natural numbers show that the number of paths from $(0, 0, 0)$ to (a, b, c) is

$$\frac{(a+b+c)!}{a!b!c!}$$

(4) If a, b, c are natural numbers with $c > a + b$, find an expression for the number of paths from $(0, 0, 0)$ to (a, b, c) such that $z > x + y$ for every point (x, y, z) on the path except for $(0, 0, 0)$. (Geometrical picture: we are looking at paths which stay above the plane $z = x + y$).

(5) We are given n letters $L_1, \ldots L_n$ and n envelopes $E_1, \ldots E_n$. How many ways are there of putting the letters in the envelopes

so that exactly one letter goes in each envelope? How many ways are there such that for every i, letter L_i does not go into envelope E_i?

(6) (Challenging) Recall that in class we defined the n^{th} *Catalan number* to be

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

or equivalently the number of paths from $(0,0)$ to (n,n) which do not go below the line $y = x$.

(a) Show that C_n is the number of sequences $(1, a_1, \ldots a_n)$ with $a_i \in \mathbb{N}$ and $a_i \leq i$ and $1 \leq a_1 \leq \ldots \leq a_i$.

(b) Given $2n$ distinct points on the circumference of a circle, C_n is the number of ways of joining them in pairs by drawing n chords of the circle, no two of which intersect. Here is a picture for the case $n = 2$.