DISCRETE MATHEMATICS 21228 — HOMEWORK 2 SOLUTIONS

JC

Due in class Wednesday September 10. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

(1) Consider a recurrence relation $a_{n+2} = Aa_{n+1} + Ba_n$ where as usual $A, B \in \mathbb{R}$. Suppose that the associated quadratic equation $\lambda^2 = A\lambda + B$ has a *repeated root*, that is to say $\lambda^2 - A\lambda - B = (\lambda - \lambda_1)^2$ for some λ_1.

(a) Show that $a_n = \lambda_1^n$ and $a_n = n\lambda_1^n$ both give solutions of the recurrence relation $a_{n+2} = Aa_{n+1} + Ba_n$.
We know that $A = 2\lambda_1$ and $B = -\lambda_1^2$. Now

$$\lambda_1^{n+2} - 2\lambda_1\lambda_1^{n+1} + \lambda_1^2\lambda_1^n = 0,$$

and also

$$(n+2)\lambda_1^{n+2} - 2(n+1)\lambda_1\lambda_1^{n+1} + n\lambda_1^2\lambda_1^n = 0.$$

(b) Find a formula for a_n where $a_0 = a_1 = 1$ and $a_{n+2} = 4a_{n+1} - 4a_n$.
The associated quadratic is $\lambda^2 - 4\lambda + 4 = 0$, which has a repeated root $\lambda = 2$ so that we have solutions 2^n and $n2^n$. As in the easier case with distinct real roots, we try to combine these two solutions into a solution $C2^n + Dn2^n$ where C and D are chosen to satisfy the initial conditions. If $a_n = C2^n + Dn2^n$ then $a_0 = C = 1$ and $a_1 = 2C + 2D = 1$ so that $C = 1$ and $D = -1/2$. So the solution is $a_n = 2^n - n2^{n-1}$.

(2) A *complex number* is an expression of the form $a + bi$ where $a, b \in \mathbb{R}$. Complex numbers are added and multiplied by the rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i.$$

Note that $i \times i = -1$.

Prove that if $a + bi$ is a nonzero comples number (that is to say at least one of the reals a and b is nonzero) there is a complex number $c + di$ such that $(a + bi) \times (c + di) = 1$, and find formulae for the reals c and d in terms of a and b.

Here are two ways of doing it:

HARD WAY: from the formulae above we need to solve the equations $ac - bd = 1$, $ad + bc = 0$. We see that

$$b = b(ac - bd) - a(ad + bc) = -d(a^2 + b^2),$$

and

$$a = a(ac - bd) + b(ad + bc) = c(a^2 + b^2).$$

So $d = -b/(a^2 + b^2)$ and $c = a/(a^2 + b^2)$.

EASY WAY: notice that $(a + bi)(a - bi) = a^2 + b^2$ so that $(a^2 + b^2)(c + di) = a - bi$, thus $d = -b/(a^2 + b^2)$ and $c = a/(a^2 + b^2)$.

Now consider the recurrence relation $a_{n+2} = 2a_{n+1} - 2a_n$ with its associated quadratic equation $\lambda^2 = 2\lambda - 2$.

(a) Show that the complex numbers $1 + i$ and $1 - i$ are roots of the quadratic equation $\lambda^2 = 2\lambda - 2$.
$(1 + i)^2 = 1 + 2i - 1 = 2i = 2(1 + i) - 2$, and $(1 - i)^2 = 1 - 2i - 1 = -2i = 2(1 - i) - 2$.

(b) Find complex numbers C and D such that $C + D = 3$ and $C(1 + i) + D(1 - i) = 5$. Compute $C(1 + i)^n + D(1 - i)^n$ for n from 0 to 5. What do you notice?
Multiplying the first equation by $1+i$, $C(1+i)+D(1+i) = 3 + 3i$, so subtracting we get $D(-2i) = 2 - 3i$. So easily $D = 3/2 + i$ and $C = 3/2 - i$.
A tedious calculation now gives $C + D = 3$, $C(1 + i) + D(1 - i) = 5$, $C(1 + i)^2 + D(1 - i)^2 = 2iC - 2iD = 4$, $C(1 + i)^3 + D(1 - i)^3 = -2$, $C(1 + i)^4 + D(1 - i)^4 = -12$, $C(1 + i)^4 + D(1 - i)^4 = -20$.
The first thing you might notice is that these are real numbers. In fact they are the first few values of the sequence given by $a_{n+2} = 2a_{n+1} - 2a_n$, $a_0 = 3$, $a_1 = 5$. So actually the same trick we used for solving recurrences when the quadratic has distinct real roots is still working, only now we need to use complex numbers.

(3) Find the power series expansion of $\frac{1}{1+x+x^2}$.

There are many ways of doing this, here is the least painful one that I found:

$$\frac{1}{1+x+x^2} = \frac{1-x}{1-x^3} = (1-x)(1+x^3+x^6+\ldots) = 1-x+x^3-x^4+\ldots$$

(4) Consider the recurrence relation $a_{n+2} = 5a_{n+1} - 6a_n$. For which choices of a_0 and a_1 do we get a sequence a_n such that $\lim_{n\to\infty} a_{n+1}/a_n = 2$? What other limits are possible and when do they occur?

By the usual argument the general solution has the form $a_n = C2^n + D3^n$. Now we can distinguish three cases.

If $C = D = 0$ then $a_n = 0$ for all n and the limiting ratio is not defined.

If $C \neq 0$ and $D = 0$ then $a_n = C2^n$ and $a_{n+1}/a_n = 2$ for all n.

If $D \neq 0$ then we can write

$$\frac{a_{n+1}}{a_n} = \frac{2C/D(2/3)^n + 3}{C/D(2/3)^n + 1},$$

which will clearly tend to 3 as n tends to infinity.

So the only case with limiting ratio 2 occurs when $C \neq 0$, $D = 0$. This happens exactly when $a_0 \neq 0$ and $a_1 = 2a_0$.