DISCRETE MATHEMATICS 21228 — HOMEWORK 2

JC

Due in class Wednesday September 10. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in blue or black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

(1) Consider a recurrence relation $a_{n+2} = Aa_{n+1} + Ba_n$ where as usual $A, B \in \mathbb{R}$. Suppose that the associated quadratic equation $\lambda^2 = A\lambda + B$ has a *repeated root*, that is to say $\lambda^2 - A\lambda - B = (\lambda - \lambda_1)^2$ for some λ_1.

(a) Show that $a_n = \lambda_1^n$ and $a_n = n\lambda_1^n$ both give solutions of the recurrence relation $a_{n+2} = Aa_{n+1} + Ba_n$.

(b) Find a formula for a_n where $a_0 = a_1 = 1$ and $a_{n+2} = 4a_{n+1} - 4a_n$.

(2) A *complex number* is an expression of the form $a + bi$ where $a, b \in \mathbb{R}$. Complex numbers are added and multiplied by the rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i.$$

Note that $i \times i = -1$.

Prove that if $a + bi$ is a nonzero comples number (that is to say at least one of the reals a and b is nonzero) there is a complex number $c + di$ such that $(a + bi) \times (c + di) = 1$, and find formulae for the reals c and d in terms of a and b.

Now consider the recurrence relation $a_{n+2} = 2a_{n+1} - 2a_n$ with its associated quadratic equation $\lambda^2 = 2\lambda - 2$.

(a) Show that the complex numbers $1 + i$ and $1 - i$ are roots of the quadratic equation $\lambda^2 = 2\lambda - 2$.

(b) Find complex numbers C and D such that $C + D = 3$ and $C(1 + i) + D(1 - i) = 5$. Compute $C(1 + i)^n + D(1 - i)^n$ for n from 0 to 5. What do you notice?

(3) Find the power series expansion of $\frac{1}{1+x+x^2}$.

(4) Consider the recurrence relation $a_{n+2} = 5a_{n+1} - 6a_n$. For which choices of a_0 and a_1 do we get a sequence a_n such that $\lim_{n\to\infty} a_{n+1}/a_n = 2$? What other limits are possible and when do they occur?