DISCRETE MATHEMATICS 21228 — HOMEWORK 1 SOLUTIONS

JC

Due in class Wednesday September 3. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

(1) Recall that the binomial coefficent $\binom{n}{k}$ ("n choose k") is given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for natural numbers k and n with $0 \leq k \leq n$.

(a) Show that when n is even the largest value of $\binom{n}{k}$ occurs when $k = n/2$.

Let $0 \leq k < n$. It is is easy to see that

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1}.$$

So

$$\binom{n}{k+1} > \binom{n}{k} \iff n-k > k+1 \iff n > 2k+1,$$

and

$$\binom{n}{k+1} < \binom{n}{k} \iff n-k < k+1 \iff n < 2k+1.$$

So for n even the coefficient $\binom{n}{k}$ increases until $k = n/2$ and then decreases again.

Note: of course you can use differences, but ratios tend to be simpler when factorials are involved.

(b) Use Stirling's formula to find an approximate expression for $\binom{n}{n/2}$, and use your expression to estimate roughly how large is the least n for which $\binom{n}{n/2} > 2^{100}$.

For convenience we let $n = 2m$ and then an approximate value is

$$\frac{\sqrt{4\pi m}(2m/e)^{2m}}{(2\pi m)(m/e)^{2m}} = 2^{2m}/\sqrt{\pi m}$$

Taking logs, we need to find the least m such that

$$2m\ln(2) - 1/2\ln(\pi m) > 100\ln(2)$$

A little work with a calculator shows that $m = 52$ and so $n = 104$.

(c) (Optional and not for credit) Find exactly the least n for which $\binom{n}{n/2} > 2^{100}$ and compare with the answer from the previous part.

$$\binom{102}{51} = 399608854866744452032002440112$$
$$< 2^{100} = 1267650600228229401496703205376$$
$$< \binom{104}{52} = 1583065848125949175357548128136$$

so the approximation is pretty good.

(2) Let k and n be natural numbers with $2k \leq n$. Given an n-element set C, how many unordered pairs $\{A, B\}$ are there such that $A, B \subseteq C$, $|A| = |B| = k$, $A \cap B = \emptyset$? How many unordered pairs of subsets such that $|A| = |B| = k$ and $|A \cap B| = 1$?

Start by counting ordered pairs of disjoint subsets with size k. If (A, B) is such a pair then there are $\binom{n}{k}$ possibilities for A, and then for any given A there are $\binom{n-k}{k}$ possibilities for B. Since we are counting unordered pairs this gives a total of

$$\frac{\binom{n}{k} \times \binom{n-k}{k}}{2}$$

For the second problem note that if $A \cap B = \{x\}$ then $A \setminus \{x\}$ and $B \setminus \{x\}$ are disjoint sets of size $k - 1$. It follows easily that the number of such pairs is

$$\frac{n \times \binom{n-1}{k-1} \times \binom{n-k}{k-1}}{2}$$

(3) Show that n^2 is $O(2^n)$ and that 2^n is not $O(n^2)$.

Notice that

$$\frac{(n+1)^2}{n^2} \leq 2$$

for $n \geq 3$. So easily $(3 + k)^2 \leq 9 \times 2^k$ for all natural k.

If $n = 3 + k$ this shows that $n^2 \leq 9 \times 2^{n-3} = 9/8 \times 2^n$ for $n \geq 3$, and so by definition n^2 is $O(2^n)$.

For the other direction we use calculus to compare the logarithms of the functions 2^x and Cx^2 where $C > 0$ is fixed. Let $f = \ln(2^x) - \ln(Cx^2) = x\ln(2) - \ln(C) - 2\ln(x)$. Then the derivative of f is $\ln(2) - 2/x$ and so the derivative is at least 0.5 for all $x \geq 20$. So $f(x) \geq f(20) + 0.5 \times (x - 20)$ for $x \geq 20$, so $f(x) > 0$ for all sufficiently large x and it follows that $2^x > Cx^2$ for all sufficiently large x.

(4) Find a family of 6 subsets of $\{1, 2, 3, 4\}$ such no member of the family is a subset of any other member. Can you find such a family of size 7?

Consider the set of all subsets of size 2. There are $\binom{4}{2} = 6$ of them.

We claim it is impossible to find 7 such sets. Suppose that F is a suitable family of size 7. Then clearly F cannot contain sets of size 0 or 4 so F consists of sets of size 1, 2, or 3. Since there are only 6 sets of size 2, F contains a set of size 1 or a set of size 3. Replacing F by the set $\{\{1, 2, 3, 4\} \setminus A : A \in F\}$ if necessary, we may assume F contains a set of size 1, without loss of generality $\{1\}$. But then all other elements of F must be subsets of $\{2, , 3, 4\}$ and this is clearly impossible.

Note: this is a special case of a general fact (Sperner's theorem) which we prove later.

(5) Find a formula for the number of sequences of 0's and 1's of length n which do not contain two successive 0's.

This is a little tricky. Let a_n be the number of such sequences ending in 0 and b_n the number ending in 1. Easily $a_{n+1} = b_n$ and $b_{n+1} = a_n + b_n$, with $a_1 = b_1 = 1$. Let $c_n = a_n + b_n$. Computing the first few values we see that $c_1 = 2$, $c_2 = 3$, $c_3 = 5$ and so on. It looks like the sequence of c_n is given by the recurrence $c_{n+2} = c_{n+1} + c_n$ (the Fibonacci recurrence). This is easily proved,

$$c_{n+2} = a_{n+2} + b_{n+2} = b_{n+1} + a_{n+1} + b_{n+1} = c_{n+1} + c_n.$$

So c_n is the $(n + 2)^{\text{th}}$ Fibonacci number.

(6) Let a, b, c be natural numbers. Prove that

$$\binom{ab}{c} = \sum_{i_1+i_2+\ldots i_b=c} \binom{a}{i_1} \times \binom{a}{i_2} \times \ldots \binom{a}{i_b}$$

Hint: interpret both sides combinatorially, or use the Binomial Theorem.

Let X be a subset of $\{1, \dots ab\}$ with c elements. If we let $X_n = X \cap \{(n-1)a + 1, \dots na\}$ then we have a bijection $X \longmapsto (X_1, \dots X_b)$ between the set of subsets of $\{1, \dots ab\}$ with c elements and the set of b-tuples $(A_1, \dots A_b)$ where $A_n \subseteq \{(n-1)a + 1, \dots na\}$ and $\sum_n |A_n| = c$. The equality between the LHS and the RHS is now clear.

Alternative proof. Consider the coefficient of $x^c y^{ab-c}$ in the expansion of $(x+y)^{ab} = ((x+y)^a)^b = (\sum_i \binom{a}{i} x^i y^{a-i})^b$.

(7) Let $f(n) = 1 + 1/2 + 1/3 + \dots 1/n$. Prove that f is $O(\ln(n))$. Is it true that $f \sim \ln(n)$?

By the kind of geometrical argument we used in class to estimate $\ln(n!)$, $f(n) \geq \ln(n+1)$ and $f(n) - 1 \leq \ln(n)$, that is $\ln(n+1) \leq f(n) \leq \ln(n) + 1$. Dividing by $\ln(n)$,

$$\frac{\ln(n+1)}{\ln(n)} \leq \frac{f(n)}{\ln(n)} \leq 1 + \frac{1}{\ln(n)}.$$

Notice that the lower bound is greater than one, while the upper bound is decreasing with limit one. Letting n tend to infinity we see that $f(n)/\ln(n)$ tends to one. So $f \sim \ln(n)$ and in particular f is $O(\ln(n))$.

(8) Consider the sequence recursively defined by $y_{n+2} = y_{n+1} \times y_n^2$ with $y_0 = y_1 = 2$. Find a formula for y_n.

Let $z_n = \ln(y_n)/\ln(2)$ so that $z_{n+2} = z_{n+1} + 2z_n$ with $z_0 = z_1 = 1$.

The equation $\lambda^2 - \lambda - 2 = 0$ has roots $\lambda = 2, -1$ and a little thought shows that therefore $z_n = (2/3)2^n + (1/3)(-1)^n$. Finally $y_n = 2^{z_n}$.