DISCRETE MATHEMATICS 21228 — HOMEWORK 1

JC

Due in class Wednesday September 3. You may collaborate but *must* write up your solutions by yourself.

Late homework will not be accepted. Homework must either be typed or written legibly in black ink on alternate lines, illegible homework will be returned ungraded (so you can rewrite it legibly).

(1) Recall that the binomial coefficent $\binom{n}{k}$ ("n choose k") is given by the formula
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
for natural numbers k and n with $0 \leq k \leq n$.
  (a) Show that when n is even the largest value of $\binom{n}{k}$ occurs when $k = n/2$.
  (b) Use Stirling's formula to find an approximate expression for $\binom{n}{n/2}$, and use your expression to estimate roughly how large is the least n for which $\binom{n}{n/2} > 2^{100}$.
  (c) (Optional and not for credit) Find exactly the least n for which $\binom{n}{n/2} > 2^{100}$ and compare with the answer from the previous part.

(2) Let k and n be natural numbers with $2k \leq n$. Given an n-element set C, how many unordered pairs $\{A, B\}$ are there such that $A, B \subseteq C$, $|A| = |B| = k$, $A \cap B = \emptyset$? How many unordered pairs of subsets such that $|A| = |B| = k$ and $|A \cap B| = 1$?

(3) Show that n^2 is $O(2^n)$ and that 2^n is not $O(n^2)$.

(4) Find a family of 6 subsets of $\{1, 2, 3, 4\}$ such no member of the family is a subset of any other member. Can you find such a family of size 7?

(5) Find a formula for the number of sequences of 0's and 1's of length n which do not contain two successive 0's.

(6) Let a, b, c be natural numbers. Prove that
$$\binom{ab}{c} = \sum_{i_1+i_2+\ldots i_b=c} \binom{a}{i_1} \times \binom{a}{i_2} \times \cdots \binom{a}{i_b}$$

Hint: interpret both sides combinatorially, or use the Binomial Theorem.

(7) Let $f(n) = 1 + 1/2 + 1/3 + \ldots 1/n$. Prove that f is $O(\ln(n))$. Is it true that $f \sim \ln(n)$?

(8) Consider the sequence recursively defined by $y_{n+2} = y_{n+1} \times y_n^2$ with $y_0 = y_1 = 2$. Find a formula for y_n.