DISCRETE MATHEMATICS 21228 — FINAL

JC

Due by midnight on Fri Dec 12. You may work on this exam during any continuous 24 hour period. You may not not collaborate but may consult with me, and may look at any books or papers which you like (please cite any source which you consult).

You should attempt *exactly two* questions from *each* of the three sections. If you attempt more than two from a section I will count your two lowest scores so attempting more than two is a bad strategy. All questions carry the same number of points.

Unless specified otherwise all graphs are finite.

1. Section A.

(1) Show that any graph with at least two vertices contains two vertices of equal degree. Describe the graphs which contain exactly one pair of vertices of equal degree.

(2) The *degree sequence* of a graph with n vertices is the sequence of length n formed by writing the degrees of the vertices in increasing order. For example the triangle has degree sequence $2, 2, 2$.

Show that for $n \geq 2$ a sequence of natural numbers $d_1 \leq d_2 \leq \ldots \leq d_n$ is the degree sequence of a tree if and only if $1 \leq d_1$ and $\sum_i d_i = 2n - 2$.

(3) Let G be a graph with e edges. Show that we can partition the vertex set into disjoint sets U and W so that $e(U, W) \geq e/2$ where $e(U, W)$ is the number of edges which join a member of U to a member of W.

2. Section B.

(4) We let $p_G(n)$ be the number of vertex colourings of G using colours $1, 2, \ldots n$. Let G be a graph, let a and b be vertices of G not joined by an edge of G. Let G_0 be the graph obtained from G by adding to it the edge ab, and let G_1 be the graph obtained by identifying a and b.

(a) Show that $p_G(n) = p_{G_0}(n) + p_{G_1}(n)$.

(b) Show that for every G, $p_G(n)$ is a polynomial in n.

(5) Show that the number of ways of arranging the elements $\{1, \dots n\}$ in an order which does not contain a descending sequence of length 3 is given by the Catalan number C_n. For example when $n = 3$ we have 5 arrangements 123, 132, 213, 231, 312 but of course we have to omit 321.

(6) Let G be the triangle graph with vertices a, b, c and edges ab, bc, ca. Find a formula for the number of walks with n steps which begin at a and end at b.

3. Section C.

(7) We construct a *random graph* with vertex set $\{1, \dots N\}$ as follows: for each pair ij with $1 \leq i < j \leq N$ toss a fair coin to decide whether ij appears as an edge in the graph. If Φ is some property of graphs then let $P_N(\Phi)$ denote the probability that a random graph on N vertices has property Φ.

Let G be an arbitary finite graph. Show that if Φ is the property of having an induced subgraph which is isomorphic to G then $P_N(\Phi) \to 1$ as $N \to \infty$.

(8) Let G be a graph with v vertices in which every vertex has degree at least $\lfloor v/2 \rfloor$. Show that G is connected.

(9) In a question on the midterm about counting lattice paths we encountered a sequence of numbers given by the following recursion: $E_0 = 1$, and then

$$E_n = \sum_{i+j+k=n-1} E_i E_j E_k.$$

Prove (by any method you like) that $E_n = \frac{1}{2n+1}\binom{3n}{n}$.