References: Ahlfors "Complex analysis", Conway "Functions of one complex variable".

1. Background

I will assume a knowledge of undergraduate algebra and analysis, plus elementary facts about the arithmetic of complex numbers. Here is a more detailed list (possibly subject to expansion and revision as the course proceeds).

1.1. **Algebra.** Key ideas: Groups, rings, fields, rings of polynomials:

Groups: A *group* is a set G equipped with an associative binary operation, which has a 2-sided identity element and 2-sided inverses. G is *abelian* if and only if the operation is commutative.

Rings: A *ring* is a set R equipped with binary operations $+$ and $\times$ such that $(R, +)$ is an abelian group, $\times$ is associative, and $\times$ distributes over $+$. R is *commutative* if and only if $\times$ is commutative, and *unital* if and only if $\times$ has an identity.

We write 0 for the $+$-identity and 1 for the $\times$-identity.

Fields: A *field* is a commutative unital ring in which $1 \neq 0$, and every nonzero element has a $\times$-inverse.

Polynomial rings: If k is a field then we make the set $k[x]$ of polynomials in one indeterminate x into a ring in the obvious way.

If $f \in k[x]$, then $f(a) = 0$ if and only if $x - a$ divides f in $k[x]$.

1.2. **Analysis.** Key ideas: The real field, absolute value, convergence, compactness.

The reals: $\mathbb{R}$ is a field under the usual $+$ and $\times$ operations.

Absolute value: The real absolute value function $x \mapsto |x|$ has the properties $|x+y| \leq |x| + |y|$, $|xy| = |x|.|y|$, $|x| \geq 0$, $|x| = 0 \iff x = 0$.

A real sequence (x_n) converges if and only if it is Cauchy. Every bounded sequence has a convergent subsequence.

A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded. If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f[K]$ is bounded, and contains its sup and its inf (that is to say, on K the function f is bounded and attains its bounds).

If f is continuous then f is bounded on $[a, b]$, and $|\int_a^b f(x)dx| \leq (b-a) \max_{x\in[a,b]} |f(x)|$.

1.3. **Complex numbers.** Key ideas: Complex numbers, the complex field, the complex plane (AKA Argand diagram), absolute value an argument of a complex number.

The complex numbers: We write a typical complex number in the form $z = a + ib$ where a, b are real. In this case we write $a = \Re(z)$, $b = \Im(z)$ (the *real and imaginary parts* of z). We always identify the real r with $r + 0i$, so that $\mathbb{R} \subseteq \mathbb{C}$. The plus and times operations are given by

$$(a_0 + ib_0) + (a_1 + ib_1) = (a_0 + a_1) + i(b_0 + b_1),$$

$$(a_0 + ib_0) \times (a_1 + ib_1) = (a_0a_1 - b_0b_1) + i(a_0b_1 + a_1b_0).$$

Note that $i^2 = -1$ and $\mathbb{R}$ forms a subfield of $\mathbb{C}$.

The complex field: $\mathbb{C}$ forms a field under the + and times operations. The inverse of $a+bi$ is $\frac{a-bi}{a^2+b^2}$. The map which takes $z = a+bi$ to $\bar{z} = a-bi$ is an automorphism of $\mathbb{C}$, called *complex conjugation.* $z + \bar{z} = 2\Re(z)$, $z - \bar{z} = 2i\Im(z)$.

The complex plane: $\mathbb{C}$ can be considered as a real vector space of dimension 2, which is isomorphic to $\mathbb{R}^2$ via the map $a + ib \mapsto (a, b)$. Complex addition corresponds to vector addition.

Absolute value: If $z = a+ib$ the *absolute value* $|z| = (a^2+b^2)^{1/2}$. $|z|^2 = z\bar{z}$. *Just as for the reals* we have $|z+w| \leq |z| + |w|$, $|zw| = |z|.|w|$, $|z| \geq 0$, $|z| = 0 \iff z = 0$.

The absolute value is also sometimes called the modulus or magnitude.

Argument: A complex number z can be expressed in the form $r\cos(\theta) + ir\sin(\theta)$ where $r = |z|$ (this corresponds to using polar coordinates in the complex plane). θ is called the *argument*. For $z \neq 0$ (that is $r > 0$) the value of θ is unique up to adding multiples of 2π, so we can choose a unique θ in $[-\pi, \pi)$ or $[0, 2\pi)$.

If we let $\mathrm{cis}(\theta) = \cos(\theta) + i\sin(\theta)$, then $\mathrm{cis}(\theta_1)\,\mathrm{cis}(\theta_2) = \mathrm{cis}(\theta_1 + \theta_2)$.

Notation: z, w are typically complex numbers with $z = x+iy$ and $w = u+iv$ for x, y, u, v real. In complex analysis w typically depends on z, so that we can view u, v as real valued functions of (x, y).

2. Sequences, series and functions

Note: This discussion runs parallel to that for real sequences/series/functions in undergraduate real analysis. So it is brief and most of the details are left to you to fill in. We are using the elementary properties of real and complex numbers from the preceding section, in particular we use the properties of the complex absolute value.

2.1. **Sequences of complex numbers.**

Key ideas: Convergence, limit, Cauchy sequence, completeness.

Convergence for a complex sequence: Let (z_n) be a complex sequence. Then $z_n \to z$ if and only if for all $\epsilon > 0$ there is N such that $|z_n - z| < \epsilon$ for all $n \geq N$.

Exercise: $z_n \to z$ for at most one z.

Notation: If $z_n \to z$ we write $z = \lim_{n\to\infty} z_n$.

Cauchy sequence: A *Cauchy sequence* is a sequence (z_n) such that for all $\epsilon > 0$ there is N such that $|z_m - z_n| < \epsilon$ for $m, n \geq N$. It is very easy to see that a convergent sequence is Cauchy.

Completeness for $\mathbb{C}$: Just like in $\mathbb{R}$ the converse holds, that is to say every Cauchy sequence is convergent (in the jargon, $\mathbb{C}$ is a *complete metric space*).

Let (z_n) be Cauchy with $z_n = x_n + iy_n$ for real x_n, y_n. Since $|z_m - z_n|^2 = |x_m - x_n|^2 + |y_m - y_n|^2$, we have $|x_m - x_n| \leq |z_m - z_n|$ and $|y_m - y_n| \leq |z_m - z_n|$. So (x_m) and (y_m) are Cauchy sequences of reals, and hence (by completeness for $\mathbb{R}$) converge with limits x and y say. We claim that if $z = x + iy$ then $z_n \to z$: to see this just note that $|z_n - z|^2 = |x - x_n|^2 + |y - y_n|^2$.

Exercise: A complex sequence (z_n) is *bounded* if and only if the real sequence $(|z_n|)$ is bounded. Prove that every bounded complex sequence has a convergent subsequence.

2.2. **Series of complex numbers.** Key ideas: Infinite series, partial sums, convergence for a series, absolute convergence.

Partial sums: Let (a_m) be a complex sequence. Just as in real analysis we give a meaning to the infinite sum $\sum_{m=0}^{\infty} a_m$ by considering *partial sums* $s_n = \sum_{m=0}^{\infty} a_m$.

Convergence: The series *converges to* a if and only if the sequence (s_n) is convergent with limit a, and in this case we write $\sum_{m=0}^{\infty} a_m = a$.

Absolute convergence: The complex series $\sum_{m=0}^{\infty} a_m$ is *absolutely convergent* if and only the real series $\sum_{m=0}^{\infty} |a_m|$ is convergent.

Since for $n < n'$ we have $|\sum_{n<i\leq n'} a_i| \leq \sum_{n<i\leq n'} |a_i|$, it follows readily from completeness that an absolutely convergent sequence is convergent. Sketch of argument: Partial sums of $\sum_{m=0}^{\infty} |a_m|$ convergent implies partial sums of $\sum_{m=0}^{\infty} |a_m|$ Cauchy implies partial sums of $\sum_{m=0}^{\infty} a_m$ Cauchy implies partial sums of $\sum_{m=0}^{\infty} a_m$ convergent.

2.3. **Complex functions I : Limits and continuity.** Key ideas: open set, limit of a function, continuity.

Open set in $\mathbb{C}$: A subset $U \subseteq \mathbb{C}$ is *open* if and only if for all $z \in U$ there is $\delta > 0$ such that $B(z, \delta) = \{w : |w - z| < \delta\} \subseteq U$.

Open sets in $\mathbb{C}$ have similar properties to open sets in $\mathbb{R}$: notably $\emptyset$ is open, $\mathbb{C}$ is open, a finite intersection of open sets is open, an arbitrary union of open sets is open.

Limits: Consider a complex-valued function f whose domain is an open set U in the complex plane.

Let $a \in U$, then $f(z) \to b$ as $z \to a$ if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that $B(a, \delta) \subseteq U$ and $f(z) \in B(b, \epsilon)$ for all $z \in B(a, \delta) \setminus \{a\}$.

In this case b is unique and we write $b = \lim_{z\to a} f(z)$.

Continuity: With the same assumptions, f is *continuous at* a if and only if $f(a) = \lim_{z\to a} f(z)$, and *continuous* if and only if it is continuous at all $a \in U$.

Exercise: Let $w = f(z)$ and as above let $w = u + iv$, $z = x + iy$ for real x, y, u, v. Then we can view u and v as functions $u(x, y)$ and $v(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}$. Prove that if $a = a_0 + ia_1$ then f is continuous at a if and only if both u, v are continuous at (a_0, a_1).

Using the last exercise or working directly from the definitions and imitating proofs from real analysis, we get that sums and products of continuous functions are continuous.

2.4. **Complex functions II: Differentiability.** Key ideas: Derivative, Cauchy-Riemann equations, harmonic functions, holomorphic functions.

Derivative: Let f be a complex valued function defined on an open set U, and let $z \in U$. Then just as in real analysis we say that f is *complex differentiable at* z if and only if the limit $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists, and in this case we write $f'(z)$ for the limiting value. f' is the *derivative* of f.

Exercise: The function $z \mapsto z^2$ is complex differentiable with derivative $2z$.

Exercise: If f is complex differentiable at z then f is continuous at z.

Example: The function $z \mapsto \bar{z}$ (complex conjugation) is NOT COMPLEX DIFFERENTIABLE.

To see this we consider the point $z = 0$ and approach it from two directions: first along the real axis and then along the imaginary axis. Let h_0 be a real variable, then $\lim_{h_0 \to 0} \frac{\bar{h}_0 - \bar{0}}{h_0} = 1$, but $\lim_{h_0 \to 0} \frac{\overline{ih_0} - \bar{0}}{ih_0} = -1$. It follows easily that the complex limit $\lim_{h \to 0} \frac{\bar{h} - \bar{0}}{h}$ does not exist, so $z \mapsto \bar{z}$ is not complex differentiable at $z = 0$.

Cauchy-Riemann: As usual let $w = f(z)$ with $w = u + iv$, $z = x + iy$, and view u, v as functions of x, y. Suppose that f is complex differentiable at an arbitrary point $z = x + iy \in \mathrm{dom}(f)$, then working as in the last example we obtain two expressions which must both be equal to the derivative of f at z, namely

$$\lim_{h_0 \to 0} \frac{u(x + h_0, y) - u(x, y) + i(v(x + h_0, y) - v(x, y))}{h_0}$$

and

$$\lim_{h_0 \to 0} \frac{u(x, y + h_0) - u(x, y) + i(v(x, y + h_0) - v(x, y))}{ih_0}.$$

The first expression is of course $u_x + iv_x$ and the second one is $(u_y + iv_y)/i = v_y - iu_y$, so equating real and imaginary parts $u_x = v_y$ and $v_x = -u_y$.

So if f is complex differentiable throughout its domain U, then the real and imaginary parts of f satisfy these equations (the *Cauchy-Riemann equations*) throughout U.

Examples continued: If f is conjugation then $u = x$, $v = -y$ so that the first C-R equation is satisfied nowhere (and conjugation is not complex differentiable anywhere). If $f : z \mapsto z^2$ then $u = x^2 - y^2$, $v = 2xy$ and the C-R equations are satisfied everywhere as expected.

Harmonic functions: Assuming that the relevant derivatives exist and mixed partials are equal (this is actually true, proof later) we get from the C-R equations that $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$, that is to say $u_{xx} + u_{yy} = 0$. Similarly $v_{xx} + v_{yy} = 0$. Functions with this property are called *harmonic* and the PDE they satisfy is the *Laplace equation*.

The property of complex differentiability is much stronger and more global in its effects than is the notion of differentiability in real analysis. To mark this difference, we will henceforth use a different term:

Holomorphic function: A complex function with domain an open set U is *holomorphic* if and only if $f'(z)$ exists for every point $z \in U$.

Digression: If you have been exposed to complex analysis before you may also have heard the term "analytic" used with the same meaning. I won't use this term (yet) for reasons to be explained in a few lectures time.

Here are some examples of the differences between real and complex differentiability (all proved later in the course):

- If f is holomorphic then f' is holomorphic.
- If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant.
- Suppose that f is holomorphic and $\{z : |z - z_0| < R\} \subseteq \mathrm{dom}(f)$. Then for any r with $0 < r < R$, the value of f at z_0 is the average (in an appropriate sense) of the value of f on the circle $\{z : |z - z_0| = r\}$.
- If $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ are holomorphic and agree on some non-empty open set (however tiny) then $f = g$.

This may leave you with the false impression that there can't be too many holomorphic functions. To dispel this impression, here are more facts proved later in the course:

- (A bit vaguely stated for the moment) If f is defined by a convergent power series then f is holomorphic.
- If U is a non-empty simply connected open set with $U \neq \mathbb{C}$ and $z_0 \in U$, then there exists a unique holomorphic f with domain U such that $f(z_0) = 0$, $f'(z_0) > 0$, and f defines a bijection between U and the open unit disk $D = \{z : |z| < 1\}$.

Note: "Simply connected" means roughly there are no holes in U.

2.5. **Sequences and series of functions.** So far our focus was on functions from $\mathbb{C}$ to $\mathbb{C}$. It's more natural to discuss convergence for sequences of functions in the context of metric spaces.

Metric spaces: A *metric space* is a set X equipped with a function d from X^2 to $\mathbb{R}$ such that for all $x, y, z \in X$ we have $d(x, y) = d(y, x) \geq 0$, $d(x, y) = 0 \iff x = y$, and $d(x, z) \leq d(x, y) + d(y, z)$.

Standard metrics on $\mathbb{C}$ and $\mathbb{R}$: both $\mathbb{R}$ and $\mathbb{C}$ are metric spaces with metric given by $d(x, y) = |x - y|$.

When we define concepts like convergence, continuity and so for metric spaces it will be clear that we are making generalisations of the definitions we gave above.

Convergence for a sequence in a metric space: Let (x_n) be a sequence from a metric space X. Then $x_n \to x$ if and only if for all $\epsilon > 0$ there is N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. $x_n \to x$ for at most one x, and in this case we write $x = \lim_{n\to\infty} x_n$.

Cauchy sequence in a metric space: A *Cauchy sequence* is a sequence (x_n) such that for all $\epsilon > 0$ there is N such that $d(x_m, x_n) < \epsilon$ for $m, n \geq N$. Convergent sequences are Cauchy.

Complete metric spaces: A metric space is *complete* if and only if every Cauchy sequence is convergent. $\mathbb{R}$ and $\mathbb{C}$ are complete, but for example $\mathbb{Q}$ (with the metric it inherits from $\mathbb{R}$) is not.

Open set in a metric space X: A subset $U \subseteq X$ is *open* if and only if for all $z \in U$ there is $\delta > 0$ such that $B(z, \delta) = \{w : d(w, z) < \delta\} \subseteq U$. $\emptyset$ is open, X is open, a finite intersection of open sets is open, an arbitrary union of open sets is open.

Continuous functions between metric spaces: Function f from space X to space Y is *continuous at* x if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that $f[B(x, \delta)] \subseteq B(f(x), \epsilon)$. It is *continuous* if and only if it is continuous at every x, or equivalently the preimage under f of every open subset of Y is an open subset of X.

Pointwise convergence for sequences of functions: Let X, Y be metric spaces, let $E \subseteq X$ and let (f_n) be a sequence of functions from E to Y. (f_n) is *pointwise convergent* if and only if $(f_n(x))$ is convergent in Y for every $x \in E$. This gives a function $f : E \to Y$, the (pointwise) limit of (f_n).

Unwrapping the definition, (f_n) converges to f pointwise if and only if for all $\epsilon > 0$ for all $x \in E$ there is N (depending on x) such that for all $n \geq N$ we have $d(f_n(x), f(x)) < \epsilon$.

Uniform convergence for sequences of functions: As the name suggests, the definition of *uniform convergence* is obtained by removing the dependence of N on x.

(f_n) *converges uniformly to* f if and only if all $\epsilon > 0$ there is N such that for all $x \in E$ and all $n \geq N$ we have $d(f_n(x), f(x)) < \epsilon$. (f_n) is *uniformly convergent* if it converges uniformly to some f.

The main point is that a uniformly convergent sequence of continuous functions is continuous.