COMPLEX ANALYSIS HW I SOLUTIONS

(1) Prove that there is no continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = z$ for all z. Hint: start with the observation that $f(\operatorname{cis}(\theta))$ is a continuous function of θ.

Replacing f by $-f$ if necessary, we may assume that $f(1) = 1$. For each θ, $f(\operatorname{cis}(\theta))$ is either $\operatorname{cis}(\theta/2)$ or $-\operatorname{cis}(\theta/2)$.

Now $f(\operatorname{cis}(\theta))$ is continuous and $\operatorname{cis}(\theta/2)$ is nonzero and continuous, so $f(\operatorname{cis}(\theta))/\operatorname{cis}(\theta/2)$ is a continuous function that assumes only values $+1$ and -1. By the Intermediate Value Theorem, it must be constant with value $+1$.

Finally $\operatorname{cis}(0) = \operatorname{cis}(2\pi) = 1$, so we should have $f(\operatorname{cis}(0)) = f(\operatorname{cis}(2\pi))$, but $\operatorname{cis}(0) = 1$ and $\operatorname{cis}(\pi) = -1$.

Cultural note: This line of thinking is leading us towards the ideas of analytic continuation and Riemann surfaces.

(2) The *upper half plane* is the set $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. $SL_2(\mathbb{R})$ is the set of 2×2 real matrices with determinant 1.

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $z \in \mathbb{H}$, then

$$A \cdot z = \frac{az+b}{cz+d}.$$

(a) Prove that $A \cdot z \in \mathbb{H}$.

Since c and d are real and not both 0, it is easy to see that $cz + d \neq 0$. Recall that $\Im(z) = \frac{z-\bar{z}}{2i}$.

$$A \cdot z - \overline{A \cdot z} = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}$$

since a, b, c, d are real and conjugation is an automorphism.

Since $ad - bc = 1$,

$$\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} = \frac{z-\bar{z}}{|cz+d|^2},$$

so that easily $A \cdot z \in \mathbb{H}$.

(b) Prove that $A \cdot (B \cdot z) = (AB) \cdot z$.

Routine calculation.

(c) Identify the set of A such that $A \cdot z = z$ for all z.

Routine calculation shows that $A = \pm I$.

(3) Let $f : U \to \mathbb{C}$ be holomorphic, where U is closed under complex conjugation, and define a function $g : U \to \mathbb{C}$ by $g(z) = \overline{f(\bar{z})}$. Prove that g is holomorphic and find the derivative of g.

For any ϵ the open disk of radius ϵ around 0 is closed under conjugation, so that easily for any function H defined in an open set containing 0

$$\lim_{h \to 0} H(h) = \lim_{h \to 0} H(\bar{h}).$$

So the derivative of g at z is the limit as $h \mapsto 0$ of

$$\frac{g(z+\bar{h})-g(z)}{\bar{h}} = \overline{\left(\frac{f(\bar{z}+h)-f(\bar{z})}{h}\right)},$$

which is easily seen (conjugation is continuous) to be $\overline{f'(\bar{z})}$.

(4) Let $a \in \mathbb{C}$ with $|a| < 1$, and let U be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$

(a) Prove that the map ϕ given by $\phi : z \mapsto \frac{z-a}{1-\bar{a}z}$ is holomorphic on U.
It is easy to see that the usual sum, product and quotient rules apply to complex derivatives, and that since $|a| < 1$ the denominator $1 - \bar{a}z$ is nonzero throughout U. So ϕ is holomorphic.

(b) Prove that ϕ is a bijection from U to U, and that ϕ^{-1} is also holomorphic on U.
Recall that $w\bar{w} = |w|^2$ for any complex number w. To show that when $|z| < 1$ we have $|\phi(z)| < 1$, we must show that

$$\frac{(z-a)(\bar{z}-\bar{a})}{(1-\bar{a}z)(1-a\bar{z})} < 1,$$

or equivalently that

$$(z-a)(\bar{z}-\bar{a}) < (1-\bar{a}z)(1-a\bar{z}).$$

Expanding the LHS and RHS we must prove that

$$|z|^2 - 2\Re(\bar{a}z) + |a|^2 < 1 - 2\Re(\bar{a}z) + |a|^2|z|^2,$$

which is clear as $|z| < 1$.
Suppose that $w = \phi(z) = \frac{z-a}{1-\bar{a}z}$. Then $w - \bar{a}zw = z - a$, so $w + a = z(1 + \bar{a}w)$, $z = \frac{w+a}{1+\bar{a}w}$. Hence ϕ^{-1} is a map of the same type as ϕ^{-1}, in particular it is holomorphic.

(5) Let U be the open unit disk, and let $f : U \to \mathbb{C}$ be a holomorphic function such that f takes only real values. Prove that f is constant. Hint: Cauchy-Riemann.

Let $f = u + iv$ as usual, where in this case $v = 0$ and $f = u$. By the CR equations $u_x = u_y = 0$ throughout U, and so easily u is constant (using the fact that U is connected).