COMPLEX ANALYSIS HW I

This homework is due by class time on Friday 24. Your solutions must be typeset (preferably in LaTeX) and submitted by email as a PDF file: your name and the homework number should appear at the top of the file and in the subject line of your message, and the filename should be "[YourAndrewID]-[number of the HW].pdf".

(1) Prove that there is no continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = z$ for all z. Hint: start with the observation that $f(\text{cis}(\theta))$ is a continuous function of θ.

(2) The *upper half plane* is the set $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. $SL_2(\mathbb{R})$ is the set of 2×2 real matrices with determinant 1.

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $z \in \mathbb{H}$, then

$$A \cdot z = \frac{az+b}{cz+d}.$$

  (a) Prove that $A \cdot z \in \mathbb{H}$.
  (b) Prove that $A \cdot (B \cdot z) = (AB) \cdot z$.
  (c) Identify the set of A such that $A \cdot z = z$ for all z.

(3) Let $f : U \to \mathbb{C}$ be holomorphic, where U is closed under complex conjugation, and define a function $g : U \to \mathbb{C}$ by $g(z) = \overline{f(\bar{z})}$. Prove that g is holomorphic and find the derivative of g.

(4) Let $a \in \mathbb{C}$ with $|a| < 1$, and let U be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$
  (a) Prove that the map ϕ given by $\phi : z \mapsto \frac{z-a}{1-\bar{a}z}$ is holomorphic on U.
  (b) Prove that ϕ is a bijection from U to U, and that ϕ^{-1} is also holomorphic on U.

(5) Let U be the open unit disk, and let $f : U \to \mathbb{C}$ be a holomorphic function such that f takes only real values. Prove that f is constant. Hint: Cauchy-Riemann.