CA LECTURE 8

SCRIBE: MATTHEW WRIGHT

New topic: Localisation (we see the geometrical reason behind this word later in the course).

Recall the "field of fractions" construction for an ID R. We let X be the set of pairs (a, b) where $a, b \in R$ with $b \neq 0$ and introduce a binary relation E in which $(a, b)E(c, d)$ iff $ad = bc$.

This is an equivalence relation: the only tricky bit is transitivity which goes as follows. Suppose that $(a_1, b_1)E(a_2, b_2)E(a_3, b_3)$, so that $a_1b_2 = a_2b_1$ and $a_2b_3 = a_3b_2$. Then easily $a_1b_2b_3 = a_2b_1b_3 = a_3b_1b_2$, so since $b_2 \neq 0$ and R is an ID we conclude that $a_1b_3 = a_3b_1$.

We then define a/b to be the E-class of (a, b) and attempt to define

$$a/b + c/d = (ad + bc)/bd, a/b \times c/d = ac/bd, 0 = 0/1, 1 = 1/1.$$

It is routine to check that these operations are well-defined and make the set F of E-classes into a field. Also the map $\phi : a \mapsto a/1$ is an injective HM from R to F.

The intuition is that F is the "least field" containing a copy of R. Explicitly it is easy to see that if $\psi : R \to F^*$ is an injective HM from R to a field F^*, then there is a unique HM $\alpha : F \to F^*$ such that $\alpha \circ \phi = \psi$. This α is given by the equation $\alpha : a/b \mapsto \psi(a)\psi(b)^{-1}$. We note that α is injective since it is a HM between fields.

This "universal property" determines the field F and the map $\phi : R \to F$ up to IM. To see this we introduce a category whose objects are injective HMs from R to fields. If $\phi_1 : R \to E_1$ and $\phi_2 : R \to E_2$ are two objects then a morphism from ϕ_1 to ϕ_2 is a map $\alpha : E_1 \to E_2$ such that $\alpha \circ \phi_1 = \phi_2$. The property of the field of fractions construction which we stated in the last paragraph just says that the map $r \mapsto r/1$ from R to F is an initial object. So by general nonsense if $\phi_1 : R \to F_1$ and $\phi_2 : R \to F_2$ are two objects with the property described in the last paragraph then there is a unique IM $\alpha : F_1 \simeq F_2$ such that $\alpha \circ \phi_1 = \phi_2$.

Now we generalise the FOF construction to a case where R is an arbitary ring and the set of 'denominators" is any multiplicatively closed $S \subseteq R$. We are looking for an initial object in the category whose objects are ring HMs $\psi : R \to T$ such that $\psi[S]$ is contained in the units of T. As usual a morphism from ψ_1 to ψ_2 is just a HM α such that $\alpha \circ \psi_1 = \psi_2$.

The set X is now the set of pairs (a, s) where $a \in R$ and $s \in S$. The equivalence relation used in the FOF construction won't work because now S may contain zero-divisors, so we define that $(a, s)E(b, t)$ iff there is $u \in S$ such that $u(at - bs) = 0$.

The following claims are easily checked:

(1) E is an equivalence relation: we will denote the class of (a, s) by a/s and the set of classes by $S^{-1}R$.
(2) If we define $+$, $\times$, 0 and 1 exactly as in the FOF case we make $S^{-1}R$ into a ring.

(3) The map $\phi : r \mapsto r/1$ is a (not necessarily injective) ring HM. For each $s \in S$ we have $1/s \times s/1 = 1$ so that $\phi[S]$ consists of units.
(4) For any ring T and any HM $\psi : R \to T$ such that $\psi[S]$ consists of units, there is a unique $\alpha : S^{-1}R \to T$ such that $\alpha \circ \phi = \psi$.
Sketch of proof: $a/s = \phi(a)\phi(s_)^{-1}$, so if α exists it is given by $\alpha : a/s \mapsto \psi(a)\psi(s)^{-1}$. Check this works, IE it is well-defined and is a HM.

Special case: if R is an ID and $0 \notin S$, we can identify $S^{-1}R$ with the subring of the FOF consisting of a/s with $a \in R$ and $s \in S$.

Dry as dust digression on extension and contraction of ideals. Let $\phi : R \to S$ be a ring HM and let J be an ideal of S, then $J^c = \phi^{-1}[J]$ is an ideal of R. As we saw in HW the map $a + J^c \mapsto \phi(a) + J$ is an injective HM from R/J^c to S/J. Let I be an ideal of R then I^e is the ideal in S generated by $\phi[I]$, a typical element is $\sum_{i=1}^n s_i\phi(a_i)$ for $s_i \in S$ and $a_i \in I$. Again there is a map $a + I \mapsto \phi(a) + I^e$.

The following are trivial:

(1) $I \subseteq I^{ec}$, $J^{ce} \subseteq J$, $I^{ece} = I^e$, $J^{cec} = J^c$.
(2) I is a contraction of something iff $I = I^{ce}$ iff $I^{ce} \subseteq I$ and J is an extension of something iff $J = J^{ce}$ iff $J \subseteq J^{ce}$.

Claim: every ideal J is $S^{-1}R$ is an extension. To see this let $I = J^c = \{a : a/1 \in J\}$, then if $a/s \in J$ we have $a/1 = a/s \times s/1 \in J$, so that $a \in I$ and $a/s = 1/s \times a/1 \in I^e = J^{ce}$.