CA LECTURE 7

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Recall that k is an algebraically closed field (ACF) iff it satisfies any of the equivalent conditions:

(1) Every $f \in k[x]$ of nonzero degree has a root.
(2) Every nonzero $f \in k[x]$ is a product of linear factors.
(3) The irreducibles in $k[x]$ are precisely the elements of degree one.

The so-called Fundamental Theorem of Algebra: $\mathbb{C}$ is an ACF.

Before proving the N-Satz we need a little more field theory.

Fact 1. *Let $k \leq l$ where k and l are fields and k is an ACF. If $a \in l$ is algebraic over k then $a \in k$.*

Proof. Let $f \in k[x]$ be nonzero with $f(a) = 0$. f is a product of linear factors so some nonzero linear polynomial in $k[x]$ has a as a zero hence $a \in k$. □

Recall that $k(x_1, \ldots x_t)$ is the field of fractions of $k[x_1, \ldots x_t]$.

Definition 1. *Let k and l be fields with $k \leq l$. Then $a_1 \ldots a_t \in l$ are* algebraically independent *over k iff for all $F \in k[x_1, \ldots x_t]$, $F(\vec{a}) = 0 \implies F = 0$.*

The next fcat is routine.

Fact 2. *Let $k \leq l$ be fields with $k \leq l$ and let $a_1 \ldots a_t \in l$ be alg ind over k. If $b \in l$ is transc over $k(a_1, \ldots a_t)$ then $a_1, \ldots a_t, b$ is alg indept over k.*

Fact 3. *Let k and l be fields with $k \leq l$ and let $a_1 \ldots a_t \in l$ be alg ind over k. Then $k(a_1, \ldots a_t) \simeq k(x_1, \ldots x_t)$ via an IM which fixes k and maps a_i to x_i.*

Proof. Consider the map $F \mapsto F(\vec{a})$ from $k[x_1 \ldots x_t]$ to l. It is a ring HM which has kernel $\{0\}$ since the a's are alg ind over k, so it's an injective HM from the ID $k[x_1, \ldots x_t]$ to l. By general nonsense about fields of fractions this HM extends uniquely to an injective HM $F/G \mapsto F(\vec{a})/G(\vec{a})$ from $k(x_1, \ldots x_t)$ to l. Clearly the range of this injective HM is exactly $k(\vec{a})$. □

IMPORTANT REMARK: By HW5 Q2 if $t \geq 1$ then in the situation of the last result $k(a_1, \ldots a_t)$ is not ring-finite over k.

Let k be any field and let $\vec{a} \in k^n$. Let $M_{\vec{a}}$ be the ideal of $f \in k[x_1, \ldots x_n]$ such that $f(\vec{a}) = 0$. We claim that $M_{\vec{a}}$ is maximal and equals $(x_1 - a_1, \ldots x_n - a_n)$. This is easy: $M_{\vec{a}}$ is the kernel of the surjective map $f \mapsto f(\vec{a})$ to the field k so $M_{\vec{a}}$ is maximal, and the other claim follows by successive division of $f \in M_{\vec{a}}$ by each $x_i - a_i$ in turn.

We prove the N-Satz in the following form:

Theorem 1. *(Hilbert Nullstellensatz) if k is an ACF and M is a maximal ideal of $k[x_1 \ldots x_n]$ then $M = M_{\vec{a}}$ for some $\vec{a}$.*

Proof. Let $l = k[x_1 \dots x_n]/M$, l is a field as M is maximal. $k \cap M = \{0\}$ so the map $a \in k \mapsto a + M$ is injective and sets up an IM between k and $k^* = \{a + M : a \in k\} \leq l$. In particular k^* is an ACF.

Let $A_i = x_i + M$. It will suffice to prove that each A_i is alg over k^*. For then each $A_i \in k^*$ so each $x_i + M = a_i + M$ for some $a_i \in k$, but then $M_{\vec{a}} = (x_1 - a_1, \dots x_i - a_i) \subseteq M$, and equality holds as $M_{\vec{a}}$ is maximal.

Clearly $l = k^*[A_1, \dots A_n] = k^*(A_1, \dots A_n)$, in particular l is ring finite over k^*. Also it is trivial to see that k^* is a Noetherian ring.

Suppose for a contradiction that at least one A_i is transc over k^*. Let N be the maximal length of an alg indept subsequence of the A_i so that $1 \leq N \leq n$. WLOG $A_1 \dots A_N$ are alg indept, and so by the maximal choice of N each A_j for $j > N$ is algebraic over $k^*(A_1, \dots A_N)$ (otherwise we we would get that $A_1 \dots A_N A_j$ was an alg indepts sequence of length $N + 1$).

Let $F_j = k^*(A_1, \dots A_j)$. For $j > N$ A_j is alg over F_N, so A_j is alg over F_{j-1}, so F_j is module finite over F_{j-1}. By a remark from last time $F_n = l$ is module finite over F_N. But as we remarked above F_N is not ring-finite over k^* because it is generated as a field by an alg indept set of elements, so we contradicted the techincal lemma from last time with $A = k^*$, $B = F_N$, $C = F_n = l$. □

Note that $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$ so the ACF hypothesis is needed.

Now for some (comparatively) easy corollaries.

Fact 4. *Let k be an ACF and let $f_1, \dots f_s \in k[x_1, \dots x_n]$ have no common zero in k^n. Then $(f_1, \dots f_s) = (1)$ in $k[\vec{x}]$.*

Proof. Otherwise extend $(f_1, \dots f_s)$ to a maximal ideal $M_{\vec{a}}$ and note that $\vec{a}$ is a common zero. □

Fact 5. *Let k be an ACF and let J be an ideal in $k[x_1 \dots x_n]$. Then $I(V(J)) = \sqrt{J}$.*

Proof. Obviously $\sqrt{J} \subseteq I(V(J))$ so suppose that $g \in I(V(J))$. Let $f_1 \dots f_s$ generate J and consider the set $f_1, \dots f_s, 1 - x_{n+1}g$ as a set in $k[x_1, \dots x_{n+1}]$. It has no common zero in k^{n+1} because g vanishes whenever all the f_i vanish, hence we can find $H_i \in k[x_1, \dots x_{n+1}]$ such that $H_1 f_1 + \dots H_n f_n + H_{n+1}(1 - x_{n+1}g) = 1$.

Now substitute $x_{n+1} = 1/g$ and multiply by a suitable power of g to see that for some k we have $g^k = \sum_i h_i f_i$ for some $h_i \in k[x_1 \dots x_n]$ and hence $g^k \in J$. □