CA LECTURE 5

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Definition 1. *An R-module M is Noetherian (N'ian) iff every submodule $N \leq M$ is fg.*

In particular a N'ian M is fg.

Definition 2. *A ring R is N'ian iff R is N'ian as an R-module, that is to say every ideal of R is fg.*

Here are some alternative characterisations:

Claim 1. *Let M be an R-module. The following are equivalent:*

(1) *M is N'ian.*
(2) *Every increasing chain $M_0 \leq M_1 \leq M_2 \leq \ldots$* stabilises, *that is there is N such that $M_n = M_N$ for $n \geq N$.*
(3) *Every non-empty set X of submodules of M has an element N which is* maximal under inclusion, *that is to say $N \in X$ and $N \subseteq N' \in X \implies N = N'$.*

Proof. 1 implies 2: $\bigcup M_i$ is a submodule, and if X is a finite generating set then we can find N such that $X \subseteq M_N$.

2 implies 3: If 3 fails we can construct by induction an infinite strictly increasing sequence of elements in X.

3 implies 1: Suppose that $N \leq M$ is not fg and then construct by induction a sequence of elements $n_i \in N$ such that $n_i \notin N_i = (n_i : i < j)_R$. The set of submodules $\{N_i : i \in \mathbb{N}\}$ has no maximal element.

□

Theorem 1. *Let M be an R-module and $N \leq M$. Then the following are equivalent:*

(1) *M is N'ian.*
(2) *Both N and M/N are N'ian.*

Proof. Suppose first that M is N'ian. Every submodule of N is a submodule of M hence fg. Every submodule of M/N has form $\bar{M}/N$ where $N \leq \bar{M} \leq M$, $\bar{M}$ is fg and (taking cosets of a finite set of generators) we see that $\bar{M}/N$ is fg.

Now suppose that both N and M/N are N'ian and let $A \leq M$. Choose finite generating sets $\{b_i\}$ for $A \cap N \leq N$ and $\{c_j + N\}$ for $(A+N)/N \leq M/N$, making sure that $c_j \in A$. Let $a \in A$ be arbitrary and express $a + N$ in terms of the cosets $c_j + N$, that is $a + N = \sum_j r_j(c_j + N)$. Now $a - \sum_j r_j c_j \in N$ and also in A, so we may express it as $\sum_i s_i b_i$. It follows that the c_j and b_i together generate A.

□

Remark: let R be a Noetherian ring and I an ideal of R. Then R/I is a Noetherian R-module and so easily is itself a Noetherian ring.

Theorem 2. *If R is N'ian then all fg R-modules are N'ian.*

Proof. Let $M = (m_1, \dots m_n)_R$ be an fg R-module. The map $(r_1, \dots r_n) \mapsto \sum_i r_i m_i$ is a surjective R-module HM from R^n to M, so M is isomorphic to R^n/N for some $N \leq R^n$.

So it suffices to show that R^n is a N'ian R-module for all n which we do by induction. The base case is just the hypothesis that R is a N'ian ring.

For the induction step: let N be the submodule of R^{n+1} consisting of elements $(r, 0, \dots 0)$ so that $N \simeq R$ and $R^{n+1}/N \simeq R^n$ as R-modules. Now use the last result. □

Theorem 3. *(Hilbert Basissatz) If R is a N'ian ring then $R[x]$ is a N'ian ring.*

Before starting proof, let us stress that $R[x]$ is NOT a N'ian R-module.

Proof. Let I be an ideal of $R[x]$. Define I_n to be the subset of R consisting of those a_n such that for some choice of $a_0, \dots a_{n-1}$ we have $\sum_{i=0}^n a_i x^i \in I$. Easily I_n is an ideal of R, and since $f \in I \implies xf \in I$ we see also that $I_n \subseteq I_{n+1}$.

Find N such that $I_n = I_N$ for $n \geq N$, and choose for each $i \leq N$ a finite set $\{f_{i,j}\}$ of polynomials in I of degree i whose leading coefficients generate I_i as an ideal of R. We claim the set of all $f_{i,j}$ generates I as an ideal of $R[x]$. We show that every $h \in I$ can be generated by induction on $deg(h)$. If $deg(h) = M \geq N$ then the leading coefficient of h is in $I_M = I_N$, so that subtracting a suitable R-linear combination of polynomials $x^{M-n} f_{N,j}$ we produce a polynomial in I of smaller degree (or zero). Similarly if $deg(h) = i < N$ then we may substract an R-linear combination of polynomials $f_{i,j}$ to produce a polynomial of smaller degree (or zero). □

(Thanks to Peter Lumsdaine and Lars Aiken for smoothing the proof I gave in class).

Now suppose that R and S are rings with $R \leq S$. We discuss two important "finiteness conditions" that S may satisfy over R.

Definition 3. *S is* module-finite *over R iff S is fg as an R-module, that is $S = (s_1, \dots s_n)_R$ for some $s_i \in S$.*

If $s_1, \dots s_n \in S$ we denote by $R[s_1, \dots s_n]$ the least subring of S containing $R \cup \{s_1, \dots s_n\}$. It is easy to see that the map $R[x_1, \dots x_n] \to R[s_1, \dots s_n]$ given by $f \mapsto f(s_1, \dots s_n)$ is a surjective HM of rings, so that $R[s_1, \dots s_n]$ is isomorphic to $R[x_1, \dots x_n]/I$ for some ideal I.

Definition 4. *S is* ring-finite *over R iff S is fg as an R-algebra, that is $S = R[s_1, \dots s_n]$ for some $s_i \in S$.*

Theorem 4. *If $R \leq S$, R is a N'ian ring and S is ring-finite over R then S is a N'ian ring.*

Proof. By repeated application of the Bassisatz $R[x_1, \dots x_n]$ is N'ian and as we saw above S is IM'ic to a quitient of this polynomial ring. □