CA LECTURE 4

SCRIBE: JONATHAN GROSS

Let R be a ring.

Definition 1. *The* ***nilradical*** *of R is $\sqrt{0}$.*

Claim 1. $\sqrt{0} = \bigcap\{P : P \text{ prime }\}$

Proof. We already saw that $\sqrt{0} \subseteq \bigcap\{P : P \text{ prime}\}$. Suppose $a \in R$ is not nilpotent, and let $S = \{1, a, a^2, \ldots\}$. Note that $0 \notin S$ and S is multiplicatively closed, so any maximal element in $\{I : I \cap S = \emptyset, I \text{ ideal}\}$ is a prime ideal not containing a. ☐

If a is a unit, then $(a) = R$ so a is not in any maximal ideal. If a is a nonunit, then $(a) \neq R$, so (a) can be extended to a maximal ideal of R.

From this, we can conclude that $\bigcup\{M : M \text{ maximal ideal}\}$ is the set of nonunits.

Definition 2. *We say R is* ***local*** *iff R has exactly one maximal ideal.*

Claim 2. *R is local iff the set of nonunits in R forms an ideal.*

Definition 3. *The* ***Jacobson radical*** *of R (denoted $\mathfrak{J}$) is the intersection of all maximal ideals.*

Claim 3. *Let M be a maximal ideal, and let $r \in R$. Then, $r \notin M$ iff there is $s \in R$ such that $rs - 1 \in M$.*

Proof. Note that R/M is a field as M is maximal, so $r \notin M$ iff $r + M \neq 0$ in R/M iff $r + M$ unit. So this is true iff there is $s \in R$ such that $1 + M = (r + M)(s + M) = rs + M$, or $rs - 1 \in M$. ☐

Now, $r \notin \mathfrak{J}$ iff there is M maximal such that $r \notin M$. From the previous lemma, $r \notin M$ iff there is $s \in R$ such that $rs - 1 \in M$. Taking the contrapositive, we see that $r \in \mathfrak{J}$ iff for all M maximal and all $s \in R$, $rs - 1 \notin M$. This is equivalent to saying that for all $s \in R$ $rs - 1$ is a unit. For cosmetic reasons we we rewrite the conclusion as $r \in \mathfrak{J}$ iff $1 + rs$ is a unit for all $s \in R$.

Definition 4. *We say that M is an R-**module** iff*

1. *$(M, +)$ is an abelian group*
2. *There is a map $R \times M \to M$ that maps $(r, m) \mapsto rm$ such that*
   - (a) $r(m_1 + m_2) = rm_1 + rm_2$
   - (b) $(rs)m = r(sm)$
   - (c) $(r_1 + r_2)m = r_1 m + r_2 m$
   - (d) $1m = m, 0m = 0$

Example: Let $\phi : R \to S$ be a ring HM. Define scalar multiplication $R \times S \to S$ by $(r, s) \mapsto \phi(r)s$. NOTE: As in this course we are assuming $\phi(1_R) = 1_S$, this makes S into an R-module.

Definition 5. *An R-**algebra** is a ring S together with a ring HM $R \to S$.*

Note: If R is a ring, then R is an R-module.

Definition 6. *Let M be an R-module. Then, $N \subseteq M$ is a* ***submodule*** *of M (we write $N \leq M$) iff*

(1) $(N, +) \leq (M, +)$
(2) $\forall r \in R, \forall n \in N, rn \in N$.

Note: The R-submodules of R are the ideals.

If $N \leq M$, then M/N has a module structure by $r(m+N) = rm + N$. This is well-defined since $m_1 + N = m_2 + N$ iff $m_1 - m_2 \in N$, so $r(m_1 - m_2) \in N$.

Definition 7. *If M, N are R-modules, then $\phi : M \to N$ is a* ***module HM*** *iff*

(1) $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$
(2) $\phi(rm) = r\phi(m)$.

First IM theorem: $im(\phi) \cong M/\ker(\phi)$.

Claim 4. *Let M be an R-module, $X \subseteq M$. The* ***least submodule of*** *M* ***containing*** *X is*

$$(X)_R =_{\text{def}} \{ \sum_{\textit{finite}} r_i x_i : r_i \in R, x_i \in X \}$$

Definition 8. *We say M is* ***finitely generated****(fg) iff there is $X \subseteq M$ finite such that $(X)_R = M$.*

Fact: There is an integral domain R and a fg R-module M such that not all submodules of M are fg

Example: Let $R = \mathbb{Z}[x_1, x_2, \ldots] = \bigcup_{i \in \mathbb{N}} \mathbb{Z}[x_1, \ldots, x_i]$. Let $M = R$, and let $N = (x_1, x_2, \ldots)_R$. Note that $M = (1)_R$, so M is f.g. However, N is not f.g. Suppose that $N = (f_1, \ldots, f_k)_R$. Choose m so large that all variables appearing in the f_is are x_j for some $j < m$. As $x_m \in N$, we have $x_m = \sum g_i f_i$ for some $g_i \in R$. Set $x_j = 0$ for $j < m$ and $x_m = 1$ to get a contradiction, as all polynomials in N have no constant term.