CA LECTURE 30

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Theorem: if R is local N'ian ring then $dim(R) < \infty$.

Running assumption for today: R is a N'ian local ring with maximal ideal I.

We will associate to such an R two other numbers $\delta(R)$ and $d(R)$ and prove that $\delta(R) \geq d(R) \geq dim(R) \geq \delta(R)$.

Last time: We were given a N'ian local R with maxl ideal I, an I-primary ideal Q, a fg R-module M and a Q-stable filtration $\{M_n\}$ of M. We showed that there is a polynomial g wuch that for large n we have $l(M/M_n) = g(n)$. The following facts about g are critical: the degree of g is bounded by the least size of a generating set for Q, and the leading term of g is independent of the choice of Q-stable filtration.

Subtle point: What do we mean by $l(M/M_n)$? Each M_n/M_{n+1} is fg as a module over the Artinian ring R/Q. hence has finite length as an R/Q-module. The R-submodules coincide with the R/Q-submodules so M_n/M_{n+1} has the same finite length as an R-module. Now M/M_n is just an R-module but by additivity its length $l(M/M_n)$ as an R-module is the sum of the lengths of the M_n/M_{n+1} (as R-modules or as R/Q-modules).

Now the natural Q-stable filtration for M is $\{Q^nM\}$. Using this we define a polynomial χ_Q^M such that $\chi_Q^M(n) = l(M/Q^nM)$ for all large n. Setting $M = R$ as well we define a polynomial χ_Q such that $\chi_Q(n) = l(R/Q^n)$ for all large n.

Lemma: The degree of χ_Q is independent of the choice of the I-primary ideal Q.

Proof: Since I is I-primary it's enough to compare χ_Q with χ_I. Since R is N'ian and $\sqrt{Q} = I$ there exists n such that $I^n \subseteq Q \subseteq I$, and so $I^{mn} \subseteq Q^m \subseteq I^m$ and hence $l(R/I^{mn}) \geq l(R/Q^m) \geq l(R/I^m)$ for all m. But now for large enough m we have $\chi_I(mn) \geq \chi_Q(m) \geq \chi_I(m)$, so by elementary considerations of the rate of growth of a polynomial function χ_Q and χ_I have the same degree.

Now define $d(R)$ to be the degree of χ_Q for Q any I-primary ideal of R. We also define $\delta(R)$ to be the least s such that some I-primary ideal has a generating set of size s.

Lemma: $\delta(R) \geq d(R)$.

Proof: By definition $d(R)$ is the degree of χ_I^R, so by results from last time it is bounded by $\delta(R)$.

Before closing the circle of inequalities we need a technical lemma:

Lemma: Let M be fg as an R-module and let $r \in R$ be such that $rm = 0 \implies m = 0$, that is to say the map $m \mapsto rm$ is injective. Let $N = rM$ and $M' = M/N$. Then $deg(\chi_Q^{M'}) < deg(\chi_Q^M)$.

Proof: We start by noting that $m \mapsto rm$ is an IM from M to N. The filtration $\{Q^nM\}$ induces filtrations $\{Q^nM \cap N\}$ and $\{Q^nM'\}$ of N and M' respectively. By a lemma from our discussion of inverse limits and completions (or directly) we can derive from the usual $0 \to N \to M \to M' = M/N \to 0$ an exact sequence

$$0 \to N/(Q^nM \cap N) \to M/Q^nM \to M'/Q^nM' \to 0.$$

Now by additivity if g is the polynomial such that $g(n) = l(N/(Q^n M \cap N))$ for all large n, then $g(n) - \chi_Q^M(n) + \chi_Q^{M'}(n) = 0$ for all large n, and so necessarily $g - \chi_Q^M + \chi_Q^{M'} = 0$. By the A-R lemma $\{Q^n M \cap N\}$ is a stable filtration of N, and so the leading term of g equals the leading term of χ_Q^N; but since $N \simeq M$ of course $\chi_Q^N = \chi_Q^M$, and so we conclude that the leading terms of g and χ_Q^M cancel and thus $\chi_Q^{M'}$ has a smaller degree.

Corollary: If r is not a zero-divisor in R then $d(R/(r)) < d(R)$.

The significance of the corollary is that it gives us a natural way of structuring an induction on $d(R)$.

Lemma: $d(R) \geq dim(R)$.

Proof: By induction on $d(R)$.

$d(R) = 0$. Then $l(R/I^n)$ is constant for large n, so $l(I^n/I^{n+1}) = 0$ for large n and thus $I^n = I^{n+1}$ for large n. In our discussion of Artinian rings we saw that any local N'ian ring where this happens is Artninian, and in particular has dimension zero.

Induction step: $d(R) > 0$. We show that every chain of prime ideals $P_0 \subsetneq P_1 \ldots \subsetneq P_d$ has $d \leq d(R)$, which suffices by the definition of $dim(R)$ as the supremum of the lengths of such chains. Since R has a unique maximal ideal I we have $P_d \subseteq I$.

Let $r \in P_0 \setminus P_1$. We let $R' = R/P_0$ and $R'' = R'/(r + P_0)$, and note that $r + P_0$ is not a zero divisor in R'. By the last corollary we conclude that $d(R'') < d(R')$.

Now R' is a N'ian local ring with maximal ideal I' where with a mild abuse of notation $I' = I/P_0$. It is routine to check that for every n, the surjective quotient map $R \to R'$ induces a surjective R-module HM $R/I^n \to R'/I'^n$; so the length of R'/I'^n as an R-module is less than or equal to the length of R/I^n as an R-module, but since the R-submodules of R'/I'^n coincide with its R'-submodules we see that the length of R'/I'^n as an R'-module is less than or equal to the length of R/I^n as an R-module. So $\chi_{I'}^{R'}(n) \leq \chi_I^R(n)$ for all large n, and hence $d(R') \leq d(R)$.

So now we know $d(R'') < d(R)$ so can apply the induction hypothesis. Clearly R'' is the quotient of R by $P_0 + (r)$, and since $P_0 + (r) \subseteq P_1$ the chain of primes $P_1 \subsetneq P_1 \ldots \subsetneq P_d$ induces a chain of primes $P''_1 \subsetneq \ldots \subsetneq P''_d$ of length $d - 1$ in R''. By induction $d - 1 \leq d(R'')$, so $d \leq d(R)$ as required.

Before we prove $dim(R) \geq \delta(R)$ we recall a few facts about primary decompositions in N'ian rings: every ideal I has an irredundant primary decomposition, and the minimal primes of the decomposition are the inclusion-minimal elements of the set of prime ideals which contain I. We call these the "minimal primes of I".

Recall also that the *height* of a prime P is the dimension of the localisation R_P, or more concretely the sup of the lengths of the chains of primes with last entry P.

Lemma: $dim(R) \geq \delta(R)$.

Proof: Let R have dimension d and note that easily I is the unique ideal of height d. We will produce by induction elements $a_1, \ldots a_d$ of I so that every prime containing $(a_1, \ldots a_i)$ has height at least i. It will follow easily that $(a_1, \ldots a_d)$ is I-primary; so we produced an I-primary ideal with $dim(R) = d$ generators and hence $\delta(R) \leq dim(R)$.

To start the induction, every prime containing the zero ideal has height at least zero. Suppose that $i \leq d$ and we have chose $a_1, \ldots a_{i-1}$. Let $P_1, \ldots P_k$ be those primes among the minimal primes of $(a_1, \ldots a_{i-1})$ which have height $i - 1$ (if there are any). Now clearly $I \nsubseteq P_i$ for all i and so by today's HW $I \nsubseteq \cup_i P_i$, and we may choose $a_d \in I \setminus \cup_i P_i$.

Now consider a prime ideal Q containing $(a_1, \dots a_d)$, we claim that the height of Q is at least i. Otherwise since Q contains $(a_1, \dots a_{i-1})$ it must be that Q has height exactly $i-1$. Now Q is a prime containing $(a_1, \dots a_{i-1})$ so it contains one of the minimal primes of $(a_1, \dots a_{i-1})$, P say. Now since P has height at least $i-1$ and Q has height $i-1$, it must be that $P = Q$. So $Q = P$ has height $i-1$ and is among the P_i, contradiction by the choice of a_d.