CA LECTURE 30

SCRIBE: LARS AIKEN

Recall the setting from last time: R is graded N'ian ring, R_0 is a subring, we have fixed homogeneous generators $h_1, \dots h_k$. M is a fg graded R-module so as we saw each M_n is fg as an R_0-module. λ is additive on fg R_0-modules. $P(M,t) = \sum_n \lambda(M_n)t^n$.

Cultural note: this is a "generating function" of the sort beloved by combinatorists.

Thm: $P(M,t)$ is of the form

$$\frac{f}{\prod_{i=1}^{k}(1 - t^{deg(h_k)})}$$

for some $f \in \mathbb{Z}[t]$.

Pf: By induction on k. If $k = 0$ then $R = R_0$, and since M is fg we see (look at grading) that $M_n = 0$ for large n. So $\lambda(M_n) = 0$ for large n and $P(M,t)$ is a polynomial.

The idea of the proof is used again below so we do a moe general version than is needed here. Let $r \in R_s$, that is r is a homogeneous element of degree s. Consider the map $\alpha_r : M \to M$ given by $m \mapsto rm$. It is linear so if $K = ker(\alpha_r) = \{m : rm = 0\}$ and $L = coker(\alpha) = M/rM$ then we get an exact

$$0 \to K \to M \to M \to L \to 0$$

A priori this is just a sequence of R-modules but easily $K = \oplus_n (K \cap M_n)$ and $L = \oplus_n M_n/rM_{n-s}$ (by convention $M_i = 0$ for $i < 0$) so K and L are graded R-modules. Breaking it up level by level we get for each n an exact sequence of R_0-modules

$$0 \to K_n \to M_n \to M_{n+s} \to L_{n+s} = M_{n+s}/rM_n \to 0$$

Since λ is additive we get

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+s}) - \lambda(L_{n+s}) = 0$$

Multiplying by t^{n+s} and summing over n we get

$$t^s P(K,t) - t^s P(M,t) + P(M,t) - P(L,t) + g = 0$$

where $g = \sum_{i<s}(\lambda(L_i)t^i - \lambda(M_i)t^i)$. Observe that by construction r "annihilates" K and L, that is $rK = 0$ and $rL = 0$.

Now let $r = h_k$ so now $s = deg(h_k)$. We observe that since R is N'ian and M is fg, each of K and L is fg as an R-module. But $R = R_0[h_1, \dots h_k]$ and h_k annihilates K and L, so actually (using the same generating sets if you like) K and L are fg as $R_0[h_1, \dots h_{k-1}]$-modules.

Solving for $P(M,t)$ and appealing to the induction hypothesis we are done.

DIGRESSION: Any rational function with integer coefficients can be written as $C\prod_{i=1}^{a}(x - a_i)^{n_i}$ where the a_i are distinct complex constants, C is a complex constant and the $n_i \in \mathbb{Z}$. If $n_i < 0$ then the rational function is said to have a pole of order $-n_i$ at $t = a_i$.

We define $d(M)$ to be 0 if $P(M,t)$ has no pole at $t = 1$ and to be the order of the pole at $t = 1$ otherwise.

Theorem: If $deg(h_i) = 1$ for all i, then there is a rational polynomial G of order $d(M) - 1$ such that $G(n) = \lambda(M_n)$ for all large n.

Proof: Let $d = d(M)$. By the last theorem $P(M,t)$ is $f/(1-t)^k$ for some integer polynomial f. Dividing by a suitable power of $1 - t$ we get $P(M,t) = g/(1-t)^d$ where $g(1) \neq 0$. Let $g = \sum_{l=0}^{N} a_l t^l$, so that $g(1) = \sum_l a_l \neq 0$.

We adopt the conventions

$$\binom{-1}{-1} = 1,$$

$$\binom{n}{-1} = 0$$

for $n \geq 1$,

$$deg(0) = -1.$$

Now

$$(1-t)^{-d} = \sum_{m=0}^{\infty} \binom{d-1+m}{d-1} t^m$$

and so equating coefficient we get that for $n \geq N$ the coefficient of t^n in $P(M,t)$ is

$$\lambda(M_n) = \sum_{l=0}^{N} a_l \binom{d-1+n-l}{d-1}$$

Each term is a polynomial in n with leading term $a_l n^{d-1}/(d-1)!$, so since $\sum_l a_l \neq 0$ we see that the RHS is $G(n)$ for G a polynomial of degree exactly $d - 1$.

Ultimate goal: if R is N;ian local with maxl ideal I then $dim(R)$ is the least size of a gnerating set for an I-primary ideal, in particular it's finite Use the "Hilbert polynomials" to build a bridge between these disparate notions.

Technical lemma: Let R be a N;ian local ring with maxl ideal I, let Q be an I-primary ideal, let M be a fg R-module with a stable Q-filtration $\{M_n\}$, and let s be the least size of a generating set for Q.

Then $l(M/M_n)$ is finite and for large n is given by $G(n)$ for G some polynomial of degree at most s. What is more the degree and leading coefficient of G are independent of the choice of filtration.

Proof: We build an associated graded ring $G_Q(R) = \oplus_n Q^n/Q^{n+1}$ and an associated graded module $G_Q(M) = \oplus M_n/M_{n+1}$.

We can easily verify: $G_Q(R)$ is N'ian, it is generated by a set of size s of homogeneous elements of degree 1, and $G_Q(M)$ is fg.

By the most recent HW R/Q is Artinian. Since each M_n/M_{n+1} is fg as an R/Q-module it has finite length. Also length is additive so we get

$$l(M/M_n) = \sum_{i<n} l(M_i/M_{i+1}).$$

What is more $l(M_i/M_{i+1})$ is given by a polynomial of degree less than s for all large i, so that easily $l(M/M_n)$ is given by a polynomial g of degree at most s for all large n.

Finally let M'_n be another stable filtration and let $l(M/M'_n) = g'(n)$ for all large n. Stable filtrations have bounded difference so there is k such that $M'_{n+k} \subseteq M_n \subseteq M'_{n-k}$ for all large n. But then $g'(n+k) \geq g(n) \geq g'(n-k)$ for all large n, so easily the polynomials g and g' have the same leading term.