CA LECTURE 3

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Recall that R/I is an ID iff I is prime and R/I is a field iff I is maximal.

Definition 1. *A* poset *is* $(\mathbb{P}, \leq)$ *where* $\leq$ *is transitive* ($a \leq b$ *and* $b \leq c$ *implies* $a \leq c$)*, reflexive* ($a \leq a$) *and antisymmetric* ($a \leq b$ *and* $b \leq a$ *implies* $a = b$)*.*

It is easy to see that any family of sets ordered by inclusion is a poset. Conversely if $\mathbb{P}$ is a poset and we define $C_a = \{b : b \leq a\}$ then the set of C_a ordered by inclusion forms a poset isomorphic to $\mathbb{P}$.

A *maximal element* in $\mathbb{P}$ is an element q such that $\forall r \; r \geq q \implies r = q$.

We say that $C \subseteq \mathbb{P}$ is a *chain* iff C is linearly ordered that is for all $a, b \in C$ we have $a \leq b$ or $b \leq a$. b is an *upper bound* for C iff $\forall a \in C \; a \leq b$.

Zorn's Lemma: If $\mathbb{P}$ is a poset such that every chain has an upper bound then for every p there is $q \geq p$ such that q is maximal.

It is important to note that not all chains are necessarily of the form

$$a_0 \leq a_1 \leq a_2 \leq \dots$$

For example let X be uncountable and let $\mathbb{P}$ be the poset of countable subsets of X ordered by inclusion. Then $\mathbb{P}$ has no maximal element even though every *countable* chain has an upper bound.

The following is easy:

Lemma 1. *Let R be a ring, $\mathcal{F}$ a nonempty family of ideals linearly ordered by inclusion. Then the union $\bigcup \mathcal{F}$ is an ideal.*

Theorem 1. *Let R be a ring, $I \neq R$. Then there is $J \supseteq I$ with J maximal.*

Proof. Let $\mathbb{P}$ be the set of all ideals J with $J \neq R$, ordered by inclusion. Clearly the maximal elements of $\mathbb{P}$ are the maximal ideals, so it suffices to show that chains in $\mathbb{P}$ have bounds. Let C be a chain. Then $1 \notin \bigcup C$ so $\bigcup C \neq R$, and as we just saw $\bigcup C$ is an ideal. $\square$

In particular if $R \neq 0$ then (0) is an ideal not equal to R so that R has at least one maximal ideal.

We now discuss ring elements which are in some sense "pathological".

We say that $a \in R$ is *nilpotent* iff there is $n > 0$ with $a^n = 0$. a is a *zero-divisor* iff $ab = 0$ for some nonzero b.

Let $a \neq 0$ be nilpotent and let $n > 1$ be least such that $a^n = 0$. Then $a^{n-1} \neq 0$ and $aa^{n-1} = 0$ so a is a zero-divisor. In particular in an ID 0 is the only nilpotent element.

If a is nilpotent then $1 + a$ is a unit with inverse $1 - a + \dots a^{n-1}$. In fact $u + a$ is a unit for all units u.

If J is an ideal then $\sqrt{J}$ is the set of a so that $a^n \in J$ for some $n > 0$, equivalently $a + J$ is nilpotent in R/J. We claim that $\sqrt{J}$ is an ideal. If $b^n \in J$ then $(ab)^n =$

$a^n b^n \in J$. Also if $a^m, b^n \in J$ then $(a+b)^{m+n-1} \in J$ (expand it out as a sum of terms $a^i b^j$ with $i+j = m+n-1$, so that $i \geq m$ or $j \geq n$).

J is *radical* iff $J = \sqrt{J}$ or equivalently iff 0 is the only nilpotent in R/J. If P is prime then R/P is an ID, so by the remarks above P must be radical. Note that $(0) \subseteq P$ and so $\sqrt{(0)} \subseteq \sqrt{P} = P$, that is any nilpotent element is in every prime ideal.

Remark: the intersection of any nonempty family of ideals is an ideal. So the ideal $\sqrt{(0)}$ of nilpotent elements is contained in the ideal formed by intersecting all prime ideals of R. We claim that these ideals are equal, which amounts to showing that for every non-nilpotent a there is a prime P with $a \notin P$.

Definition 2. *Let R be a ring. $S \subseteq R$ is* multiplicatively closed *iff $1 \in S$ and $\forall a, b \in S$ $ab \in S$.*

Remark: it is allowed that $0 \in S$ but this will be a pathological case.

We can use the next result to manufacture prime ideals. If $S = \{1\}$ it is just the trick for producing maximal ideals we already discussed.

Theorem 2. *Let S be multiplicatively closed with $0 \notin S$ and let $\mathbb{P}$ be the set of ideals disjoint from S. Then $\mathbb{P}$ is nonempty, $\mathbb{P}$ is closed under unions of chains, and any maximal element of $\mathbb{P}$ is prime.*

Proof. Since $0 \notin S$, $(0) \in \mathbb{P}$. Clearly it is closed under union of chains. Let I be maximal. Then $1 \notin I$ as $1 \in S$, so $I \neq R$. Also if $a \notin I$ then $I \subsetneq I + (a)$, so by maximality $I + (a)$ meets S.

So if $a, b \notin I$ we may choose $s, t \in S$ with $s \in I + (a)$ and $t \in I + (b)$. Then $st \in I + (ab)$, and since $st \in S$ while I is disjoint from S we see that $ab \notin I$. □