CA LECTURE 28

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Another Krull corollary: If R is N'ian local with unique maximal ideal M then $\cap_n M^n = \{0\}$.

Proof: Apply Krull to the map $R \to \hat{R}$ and note that $1 + M$ consists of units.

Now for a couple of easy remarks about the $\{G_n\}$ topology on a group G.

Remark 1: G is Hausdorff (T_2) in that topology iff $\cap_n G_n = \{0\}$.

Proof: As we saw G is T_2 iff $\{0\}$ is closed. Now easily $x \in \cap_n G_n$ iff every open set which contains x also contains 0, that is $x \in \overline{\{0\}}$. (Recall that in any space if we intersect all the closed sets containing some A we get $\bar{A}$, which is the least closed set containing A).

Remark 2: $\hat{G}$ is always Hausdorff in the $\{\hat{G}_n\}$ topology.

Proof: Let (g_n) be a Cauchy sequence whose class is in $\cap_n \hat{G}_n$, then it is easy to see that for any j we have $g_n \in G_j$ for all large j, so (g_n) is in the zero class. Now appeal to the precding remark.

Ultimate goal for this lecture: show that if R is N'ian, I is any ideal and $\hat{R}$ is the completion wrt the I-adic topology then $\hat{R}$ is N'ian. We approach this by an indirect route: ultimately we will use the fact that $\hat{I}^n/\hat{I}^{n+1} \simeq I^n/I^{n+1}$ to get a handle on the structure of $\hat{R}$.

Definition: a *filtered group* is a group A together with a filtration $\{A_n\}$. If A and B are filtered groups then a *HM of filtered groups* from A to B is a group HM $\phi : A \to B$ such that $\phi[A_n] \subseteq B_n$.

Remark: Such a ϕ induces maps $\alpha_n : A/A_n \to B/B_n$ given by $\alpha_n : a + A_n \mapsto \phi(a) + B_n$, and a map $\hat{\phi} : \hat{A} \to \hat{B}$. $\hat{\phi}$ can be seen two ways: in the cauchy sequence picture it maps the class of (a_n) to the class of $(\phi(a_n))$, in the inverse limit picture the α_n comprise a HM of inverse systems from the system $\vec{A} = A/A_0 \leftarrow A/A_1 \dots$ defining $\hat{A}$ to the system $\vec{B} = B/B_0 \leftarrow B/B_1 \dots$ defining $\hat{B}$, and $\hat{\phi}$ is the image of this HM under the inverse limit functor.

You could see it this way; we have a functor (which we have implicitly just described) from the category of filtered groups to the category of inverse systems and then the inverse limit functor from the category of inverse systems to the category of groups. The completion functor from filtered groups to groups is the composition of those functors.

Now we introduce another functor from filtered groups to groups, which maps a filtered group to the "associated graded group" defined as follows: $G(A) = \oplus_n G_n(A)$ where $G_n(A) = A_n/A_{n+1}$. If ϕ is a HM of filtered groups then ϕ induces $G_n(\phi) : G_n(A) \to G_n(B)$ by $a + A_{n+1} \mapsto \phi(a) + B_{n+1}$ and then as you expect $G(\phi) : (x_n) \in G(A) \mapsto (G_n(\phi)(x_n)) \in G(B)$.

The key idea in today's proof is that this functor is more tractable than the completion functor (roughly because direct sums are simpler than inverse limits, after all they are finitary and all coordinates are independent).

DIGRESSION: We digress briefly into a discussion of when the map $\hat{\phi}$ is surjective. I was a little too glib in class when I said it would suffice for all the α_n to be surjective, the real situation is trickier.

General case: Suppose we have inverse systems $\vec{C} = C_0 \leftarrow C_1 \ldots$ and $\vec{D} = D_0 \leftarrow D_1 \ldots$ and a HM $\{\beta_n : C_n \to D_n\}$ of inverse systems, with the property that all the β_n are surjective. Then it may happen that the induced map from $\varprojlim C_n$ to $\varprojlim D_n$ is not surjective: if we consider the problem of inductively constructing a preimage (c_n) for some (d_n) in the inverse limit of $\vec{D}$, then (even if $\vec{C}$ is surjective) we may get stuck when trying to satisfy the conditions $\beta_{n+1} : c_{n+1} \mapsto d_{n+1}$ and $\pi_n^C : c_{n+1} \mapsto c_n$.

This is a situation where abstract nonsense is actually very helpful. Observe that for each index n if $K_n = ker(\beta_n)$ we have a short exact sequence $0 \to K_n \to C_n \to D_n \to 0$. As usual the restriction of π_n^C induces a map $K_{n+1} \to K_n$ so we can form an inverse system $\vec{K}$ and a short exact sequence of inverse systems $0 \to \vec{K} \to \vec{C} \to \vec{D} \to 0$. As we showed in geeneral this just lets us conclude that $0 \to \varprojlim \vec{K} \to \varprojlim \vec{C} \to \varprojlim \vec{D}$ is exact, but if the system $\vec{K}$ is surjective then in fact $0 \to \varprojlim \vec{K} \to \varprojlim \vec{C} \to \varprojlim \vec{D} \to 0$ is exact, so in particular $\varprojlim \beta_n$ is surjective.

Remark: A direct argument or the same game with diagrams and exact sequences should convince you that all β_n being injective suffices for the inverse limit map of the β_n to be injective. HERE ENDETH THE DIGRESSION.

Theorem: if $G(\phi)$ is surjective (resp injective) than $\hat{\phi}$ is surjective (resp injective).

Start by noting that (trivially!) $G(\phi)$ is surjective (resp injective) iff all the $G_n(\phi)$ are surjective (resp injective).

Now we can form a diagram with exact rows as follows: the map $A_n/A_{n+1} \to A/A_{n+1}$ is inclusion, the map $A/A_{n+1} \to A/A_n$ is the usual map from the inverse system defining $\hat{A}$ and similarly for the second row.

$$\begin{tikzcd}
0 \arrow[r] & A_n/A_{n+1} \arrow[r] \arrow[d, "G_n(\phi)"] & A/A_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & A/A_n \arrow[r] \arrow[d, "\alpha_n"] & 0 \\
0 \arrow[r] & B_n/B_{n+1} \arrow[r] & B/B_{n+1} \arrow[r] & B/B_n \arrow[r] & 0
\end{tikzcd}$$

By a result from our excursion into homological algebra we can build an exact

$$0 \to ker(G_n(\phi)) \to ker(\alpha_{n+1}) \to ker(\alpha_n) \to coker(G_n(\phi)) \to coker(\alpha_{n+1}) \to coker(\alpha_n) \to 0$$

Note that $A_0 = A$ and $B_0 = B$ so that $\alpha_0 : 0 \to 0$ and $ker(\alpha_0) = coker(\alpha_0) = 0$. If $G(\phi)$ is surjective then each $G_n(\phi)$ is surjective, so each $coker(G_n(\phi)) = 0$ and we get for every n an exact sequence

$$0 \to coker(\alpha_{n+1}) \to coker(\alpha_n) \to 0$$

By an easy induction $coker(\alpha_n) = 0$ and α_n is surjective. What is more

$$ker(\alpha_{n+1}) \to ker(\alpha_n) \to 0$$

is exact so $ker(\alpha_{n+1}) \to ker(\alpha_n)$ is surjective. By the digression we see that $\hat{\phi} = \varprojlim \alpha_n$ is surjective. The proof for the case when $G(\phi)$ is injective is easier.

Now we introduce an extra layer of complexity by looking at associated graded rings and modules. Start with rings: let R be a ring (not necessarily N'ian!) and I an ideal. We have a filtration $\{I^n\}$ where by convention $I^0 = R$, so forgetting for a moment about multiplication we can form the associated graded group $G_I(R) = \oplus_n I^n/I^{n+1}$ as above. We want to make this into a graded ring.

We note that if $a \in I^n$ and $b \in I^m$ then the coset $ab + I^{m+n+1}$ depends only on the cosets $a + I^{n+1}$ and $b + I^{m+1}$, so we get a well-defined multiplication map from $I^n/I^{n+1} \times I^m/I^{m+1}$ to I^{m+n}/I^{m+n+1}. We extend this to $G_I(R)$ in the natural way defining

$$(x_n) \times (y_m) = (z_k)$$

where $z_k = \sum_{n+m=k} a_n b_m$ and of course $x_n \in I^n/I^{n+1}$, $y_m \in I^m/I^{m+1}$ and the product $x_n y_m$ is formed as above. It is routine to check this works.

Note: in this construction it is important to keep clear the distinction between $a \in I^n$, the coset $a + I^{n+1} \in I^n/I^{n+1}$ and the element of $G_I(R)$ which has $a + I^{n+1}$ at coordinate n and zeroes elsewhere. We will write this last object as a^* or possibly as a^{*n} when n is not clear from context:

Note: The subset of $G_I(R)$ consisting of elements a^{*0} is a subring and is IMic to R/I. We call this subring R^*.

We claim that if R is N'ian then so is $G_I(R)$. In fact let $b_1, \dots b_m \in I = I^1$ be generators of I. We claim that the corresponding elements b_j^{*1} generate $G_I(R)$ as a ring over R^*. To see this proceed in steps: every element of I has form $\sum_j r_j b_j$ so every element of I/I^2 has form $\sum_j r_j(b_j + I^2)$ so every element of I^{*1} has form $\sum_j r_j^{*0} b_j^{*1}$. Now go by induction (being sure to keep track of the grading) to show that if $a \in I^n$ then $a^{*n} \in R^*[b_1^*, \dots b_j^*]$. We've shown $G_I(R)$ is ring finite over R, now appeal to the Basissatz.

Now let M be an R-module with I-filtration $\{M_n\}$ and define a group $G_I(M) = \oplus M_n/M_{n+1}$. We aim to make this into a graded R-module. Start by noting that $I^a M_b \subseteq M_{a+b}$, so that both $I^{a+1} M_b$ and $I^a M_{b+1}$ are subsets of M_{a+b+1}. It follows that if $r \in I^a$ and $m \in M_b$ then $rm \in M_{a+b}$ and the coset $rm + M_{a+b+1}$ depends only on $r + I^{a+1}$ and $m + M_{b+1}$. Now we define the scalar multiplication in a very similar way to the multiplication in $G_I(R)$. We also adopt the convention that if $m \in M_n$ then m^* is the element of $G_I(M)$ with $m + M_{n+1}$ at coordinate n and zeroes elsewhere.

Important remark: if R is N'ian, I is an ideal, M is fg **and** $\{M_n\}$ is stable then $G_I(M)$ is a fg (and hence N'ian) $G_I(R)$-module. Here is a sketch of the argument. Choose n so large that $M_{m+1} = IM_m$ for $m \geq n$. Each M_n is fg (after all M is an fg module over a N'ian ring) so we may choose finite generating sets X_j for M_j where $j \leq n$. Now we argue that $\bigcup_j X_j^{*j}$ generates $G_I(M)$, showing that for all m the set M_m^{m*} is in the span. This is easy for $m \leq n$. When $m > n$ we use the fact that $M_m = IM_{m-1}$, and that we have available coefficients r^{*1} for $r \in I$.

Now we are ready for the main technical result.

Theorem: let R be a ring, I an ideal, R complete in the I-adic topology, and M an R-module with a filtration M_n such that $\cap_n M_n = \{0\}$ (that is it is Hausdorff in the $\{M_n\}$-topology. If $G_I(M)$ is fg as a $G_I(R)$-module then M is fg as an R-module. What is more, if $G_I(M)$ is N'ian as a $G_I(R)$-module then M is N'ian as an R-module.

Proof: Suppose we are given a finite generating set. Each element is a finite sum of "homogeneous" elements of form m^{*j}, so we may as well assume that the generating set consists of such elements. Explicitly we fix $m_i \in M_{n(i)}$ where $1 \leq i \leq m$ such that the elements $m_i^{*n(i)}$ are a generating set.

Let's parse this statement. It amounts to saying that for every n and and every $m \in M_n$, m^{*n} is a $G_I(R)$-linear combination of the $m_i^{*n(i)}$. By considerations of grading we may as well assume that in this linear combination the only relevant i are those with $n(i) \leq n$ and the coefficient of $m_i^{*n(i)}$ is of the form $r^{*n-n(i)}$ for some $r \in I^{n-n(i)}$.

Now let $F = R^m$ and define $\phi : F \to M$ by $\phi : (r_1, \dots r_m) \to \sum_i r_i m_i$. We will show R is surjective. To this end we filter F via F_i where F_i is the set of tuples $(r_1, \dots r_m)$ such that $r_j \in I^{i-n(j)}$ for all j such that $n(j) \leq i$. This is a stable I-filtration and (by the considerations of the preceding paragraph) has the nice property that $G(\phi) : G_I(F) \to G_I(M)$ is surjective. A little thought shows that a sequence of m-tuples in F is Cauchy for the F_i-topology iff on each of the m coordinates we have a Cauchy sequence for the I-topology, and since R is complete we see that F is also complete.

Since $G(\phi)$ is surjective, $\hat{\phi} : \hat{F} \to \hat{M}$ is surjective. So we have the diagram

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d] & M \arrow[d] \\ \hat{F} \arrow[r, "\hat{\phi}"] & \hat{M} \end{tikzcd}$$

where $F \to \hat{F}$ is an IM, $M \to \hat{M}$ is injective (recall the kernel is $\cap_n M_n$ which is $\{0\}$ by hypothesis) and $\hat{\phi}$ is surjective. A short diagram chase shows ϕ is surjective.

Finally suppose that $G_I(M)$ is a N'ian $G_I(R)$-module. Let $M' \leq M$ and consider the filtration $\{M'_n = M_n \cap M'\}$. It is routine to check that the map $a + M'_{n+1} \mapsto a + M_{n+1}$ from M'_n/M'_{n+1} to M_n/M_{n+1} is injective so we may form $G_I(M')$ and regard it as a submodule of $G_I(M)$. By hypothesis it is fg as a $G_I(R)$-module so that by the first part of the result M' is fg as an R-module, establishing that M is N'ian.

Theorem: Let R be N'ian and I an ideal of R, then $\hat{R}$ is N'ian.

Proof: Since $\hat{I}^n/\hat{I}^{n+1} \simeq I^n/I^{n+1}$, we see that $G_{\hat{I}}(\hat{R}) \simeq G_I(R)$. In particular $G_{\hat{I}}(\hat{R})$ is a Noetherian ring.

Now consider $\hat{R}$ as a filtered $\hat{R}$-module with the $\hat{I}$-topology. We know that $\hat{I}^n = \widehat{I^n}$ so that $\hat{R}$ is both complete and Hausdorff in this topology, Now apply the preceding result with $\hat{R}$ in the place of both M and R, to conclude that $\hat{R}$ is N'ian as a $\hat{R}$-module so is a N'ian ring.