CA LECTURE 27

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WEAKEN THE RUNNING ASSUMPTIONS FOR A BIT: R IS JUST A RING I IS A FIXED IDEAL AND ALL COMPLETIONS ARE IN THE I-ADIC TOPOLOGY

Easy to see that $\hat{R}$ is a ring under the obvious operations, which can be defined either in the Cauchy sequence picture or the inverse limit picture (in either case operations are coordinatewise, with the added wrinkle that in the Cauchy sequence picture we need to check we are getting well-defined operations on equivalence classes of Cauchy sequences).

Now since $\hat{I}$ is an R-module we may define $\hat{I}$ which is an ideal of $\hat{R}$. Note that the I-topology on I coincides with the topology inherited from R with its I-topology. In the Cauchy sequence picture elements of $\hat{I}$ are represented by classes of Cauchy sequences from I, in the $\varprojlim R/I^n$ picture by sequences (r_n) with $r_1 = 0 + I \in R/I$.

Now by general theory of completions we know that $\hat{R}$ is complete (equal to its own completion) when topologised using $\{\widehat{I^n}\}$. This is because by that same general theory we have $\widehat{I^n}/\widehat{I^{n+1}} \simeq I^n/I^{n+1}$ (this becomes very easy to see if you look at things in the inverse limit picture).

But of course we may also choose to topologise and complete $\hat{R}$ using $\{\hat{I}^n\}$. This is more mysterious. It is not too hard to see that $\hat{I}^n \subseteq \widehat{I^n}$ (an n-fold product of cauchy sequences from I gives a Cauchy sequence from I^n) but they won't be equal in general.

It is easy to see that $\hat{M}$ has the structure of a $\hat{R}$-module. To be explicit we define scalar multiplication so that the class of the cauchy sequnce (r_n) times the class of (m_n) is the class of $(r_n m_n)$. There is a natural map $M \to \hat{M}$, where $m \mapsto \hat{m} = [(m, m, m, \ldots)]$ which is an R-module HM.

Moreover if $\alpha : M \to N$ is an R-module HM then $\alpha[I^n M] \subseteq I^n N$ so that any cauchy sequence (m_n) maps to a cauchy sequence $(\alpha(m_n))$. Easy to see we can define $\hat{\alpha} : \hat{M} \to \hat{N}$ and that we have defined a functor from R-modules to $\hat{R}$-modules.

The following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d] & N \arrow[d] \\ \hat{M} \arrow[r, "\hat{\alpha}"] & \hat{N} \end{tikzcd}$$

We claim that if α is surjective then $\hat{\alpha}$ is surjective also. Note that routinely $\alpha[I^i M] = I^i N$. Now let (n_i) be a cauchy sequence in N, thinning out (as in the proof that the inverse limit picture is the same as the cauchy sequnce picture) we may as well assume that $n_i - n_{i+1} \in I^i N$ for all i. Choose m_0 such that $\alpha(m_0) = n_0$ and then proceed inductively by choosing $x_i \in I^i M$ with $\alpha(x_i) = n_{i+1} - n_i$ and then letting $m_{i+1} = m_i + x_i$.

Now we may also define an R-bilinear map from $\hat{R} \times M$ to $\hat{M}$ by mapping $(r_n), m$ to $(r_n m)$ and checking this respects equivalence of cauchy sequences (or just compose $M \to \hat{M}$ and scalar multiplication as above). As usual this factors through an R-linear map from $\hat{R} \otimes_R M$ to $\hat{M}$. We will use this map to analyse the structure of $\hat{M}$, the slogan might be "reduce problems about cauchy sequences in M to problems about cauchy sequences in R".

In general the map $\hat{R} \otimes_R M \to \hat{M}$ won't be either injective or surjective. But in a very simple case it is always an IM. Start by noting that $\widehat{M \oplus N} \simeq \hat{M} \oplus \hat{N}$ as $\hat{R}$-modules, because a pair of cauchy sequences is really the same thing as a cauchy sequence of pairs. So easily for any integer m we have $\widehat{R^m} \simeq \hat{R}^m$ Then using standard facts about tensor products (including the fact that tensor product distributes over finite direct sums, which is an easy exercise) we have

$$\hat{R} \otimes R^m \simeq (\hat{R} \otimes R)^m \simeq \hat{R}^m \simeq \widehat{R^m},$$

in summary $\hat{R} \otimes F \simeq \hat{F}$ when $F = R^m$.

Theorem: if M is fg then $\hat{R} \otimes_R M \to \hat{M}$ is surjective, and if in addition R is N'ian then it's an IM.

Proof: since M is fg we have for some m a surjective HM $F = R^m \to M$, so if $N \leq M$ is the kernel we have a short exact $0 \to N \to F \to M \to 0$. Tensoring with $\hat{R}$ and using an old result about tensor products and exact sequences we get an exact $\hat{R} \otimes N \to \hat{R} \otimes F \to \hat{R} \otimes M \to 0$. We may also apply the hat functor and get a (not necessarily exact) $0 \to \hat{N} \to \hat{F} \to \hat{M} \to 0$ where we note that actually the map $\hat{F} \to \hat{M}$ is surjective.

Now we use the maps $\hat{R} \otimes K \to \hat{K}$ for $K = N, F, M$ to get a commutative

$$\begin{tikzcd}
 & \hat{R} \otimes N \arrow[r] \arrow[d, "\gamma"] & \hat{R} \otimes F \arrow[r] \arrow[d, "\beta"] & \hat{R} \times M \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \hat{N} \arrow[r] & \hat{F} \arrow[r, "\delta"] & \hat{M} &
\end{tikzcd}$$

where β is an IM, δ is surjective and the top row is exact. By an old HW we see that α is surjective.

If R is N'ian then since F is fg, N is fg so that a) by the Artin-Rees argument from last time the sequence $0 \to \hat{N} \to \hat{F} \to \hat{M} \to 0$ is exact AND b) by the first part of the theorem with N in place of M, the map γ is surjective (I forgot this point in lecture I think). Now by appealing to that old HW again we see α is injective.

It's worth recording an easy corollary: if M is fg then $\hat{M}$ is the image of the map $\hat{R} \otimes_R M \to \hat{M}$, which is to say the $\hat{R}$-submodule generated by the image of the map $M \to \hat{M}$ (We will often write this as $\hat{M} = \hat{R}M$ which is formally wrong but suggestive).

RUNNING ASSUMPTION IS NOW (RE)STRENGTHENED SO THAT R IS NOETHERIAN, I FIXED IDEAL AND ALL COMPLETIONS IN THE I-ADIC TOPOLOGY.

Let J be any ideal of R, then J is fg as an R-module so that abusing notation as above $\hat{J} = \hat{R}J$, or if you prefer $\hat{J}$ is the extension of J in $\hat{R}$. In particular $\hat{I} = \hat{R}I$ and so $\hat{I}^n = \hat{R}I^n = \widehat{I^n}$. So in particular

$$\hat{I}^n/\hat{I}^{n+1} = \widehat{I^n}/\widehat{I^{n+1}} \simeq I^n/I^{n+1},$$

and so $\hat{R}$ is actually complete in the $\hat{I}$-adic topology (which is now seen to coincide with the topology from the filtration $\{\widehat{I^n}\}$. Finally if $a \in \hat{I}$ then the power series

$$1 - a + a^2 - a^3 \dots$$

converges by the completeness of $\hat{R}$ in the $\hat{I}$-adic topology, and it is easy to see that it converges to a multiplicative inverse for $1 + a$ (convergence here means, as you might hope, that the sequence of partial sums approaches some limit). So $\hat{I}$ is contained in the Jacobson radical of $\hat{R}$.

To finish we analyse the kernel of $M \to \hat{M}$ for M fg. Note that for any module the kernel is $\bigcap_n I^n M$.

Theorem (Krull): For M fg the kernel of $M \to \hat{M}$ is $\{m : \exists a \in I \ (1-a)m = 0\}$.

Proof: If $m = am$ for $a \in I$ then since $m = a^i m$ for all i, we have $m \in \bigcap_n I^n M$.

For the converse let $E = \bigcap_n I^n M$. By the A-R lemma the I-adic topology on E equals the topology from the filtration $\{I^n M \cap E\}$, but since $E = E \cap I^n M$ for all n the only open sets in the latter topology are $\emptyset, E$ (it is *indiscrete*). Since IE is an open set in the I-adic topology on E, $IE = E$, and so (by the lemmas leading up to Nakayama) there is $a \in I$ with $(1-a)E = 0$.

Corollary: If R is a Noetherian ID and $I \neq R$ then $\bigcap_n I^n = \{0\}$.