CA LECTURE 25 (SPECIAL HALLOWEEN EDITION)

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Conclusion of proof from last time: we want to show that $R = R_0[s_1, \dots s_m]$ where the s_i generate R_+ as an ideal of R. Show by induction that $R_i \subseteq R_0[s_1, \dots s_m]$ where the $i = 0$ case is none too hard. Suppose that $n > 0$ and $r \in R_n$, so $r \in R_+$ and thus $r = \sum_j r_j s_j$ for some $r_j \in R$.

Now recall we are in a graded ring so each coefficient r_j is a finite sum of "components" from the various subgroups R_i.

Since the s_j are in R_+ the equation must remain true if we discard the components of each r_j which come from R_i with $i \geq n$. But then by induction we have $r = \sum_j r_j s_j$ with each $r_j \in R_0[s_1, \dots s_m]$ and so we are done.

Definition: Let G be a group and G_n, G'_n two decreasing sequences of subgroups. They have *bounded difference* iff there is k such that $G_{n+k} \subseteq G'_n$ and $G'_{n+k} \subseteq G_n$ for all n.

Remark: by a recent homework this implies (but will in general be stronger than) the condition that the two topologies induced by the sequences should coincide.

Remark: this is an equivalence relation on sequences.

Definition: let M be an R-module. A *filtration* of M is a decreasing sequence M_n of R-submodules with $M_0 = M$. If I is an ideal of R the filtration is an *I-filtration* if $IM_n \subseteq M_{n+1}$ for all n, and is a *stable I-filtration* iff in addition $IM_n = M_{n+1}$ for all large n.

Remark; The defining sequence $I^n M$ for the I-topology on M is an I-stable filtration.

Lemma: Two stable I-filtrations of M have bounded difference.

Proof: Let M_n be stable. It's enough to show that M_n and $I^n M$ have bounded difference. By an easy induction $I^n M_0 = I^n M \subseteq M_n$. If $M_{n+1} = IM_n$ for all $n \geq k$ then by another easy induction $M_{k+j} = I^j M_k \subseteq I^j M$.

Motivating goal: when $M \leq N$ are R-modules we wish to compare the topologies on M given by $I^n M$ and $I^n N \cap M$. We do this via the Artin-Rees lemma.

Given a ring R and ideal I we construct a graded ring R^*. $R_n = I^n$, $R^* = \oplus_n R_n$ and we define the ring operations as dictated by distriobutivity and gradedness: explicitly

$$(a_n) + (b_n) = (c_n),$$

and

$$(a_n) \times (b_n) = (\sum_{i+j=n} a_i b_j)_n.$$

If M is an R-module with an I-filtration M_n then we define a graded R^*-module M^* with underlying set $\oplus_n M_n$ and the "obvious" operations.

Lemma: If R is N'ian then R^* is N'ian.

Proof: Enough to show that it is ring-finite over R_0. I is fg as an ideal so let I be generate as an ideal by $r_1, \dots r_m$. Define r_i^* to be the corresponding member of

R_1, that is $(r_i^*)_1 = r_i$ an $(r_i^*)_j = 0$ for $j \neq 1$. Verify that the r_i^* generate R^* as a ring over R_0.

Lemma: Let R be N'ian, M an fg R-module and M_n an I-filtration of M. Then the filtration M_n is stable iff M^* is a fg R^*-module.

Proof: For each n define Q_n to be the subset of M^* consisting of (m_i) such that $m_i \in M_i$ for $i < n$, and $m_i \in I^{n-i} M_n$ for $i \geq n$.

Since R is N'ian each M_n is fg as an R-module, so easily each Q_n is fg as an R^*-module. Also easy to see that $M^* = \bigcup_n Q_n$.

Now we just observe that (using the N'ian property of R^* here) M^* is fg iff the sequence of Q_n is eventually constant iff $Q_n = M^*$ for some n iff the filtration M_n is stable.

Lemma (Artin-Rees): Let R be Noetherian, N a fg R-module and $M \leq N$. Let N_n be an I-stable filtration of N. Then $N_n \cap M$ is an I-stable filtration of M.

Proof: Define graded R^*-modules M^* and N^* using the filtrations $N_n \cap M$ and N_n. It is easy to see that $M^* \leq N^*$. Now since R^* is N'ian we see that N_n stable implies N^* fg implies N^* N'ian implies M^* fg implies $N_n \cap M$ stable.

Corollary: the filtrations $I^n M$ and $I^n N \cap M$ have bounded difference and in particular they induce the same topology.

UNTIL FURTHER NOTICE: we fix a N'ian ring R and an ideal I. For any R-module M, $\hat{M}$ is the completion wrt the I-adic topology.

Corollary: if A, B, C are fg R-modules and $0 \to A \to B \to C \to 0$ is exact then the corresponding sequence $0 \to \hat{A} \to \hat{B} \to \hat{C} \to 0$ is exact.

Proof: we proved this last time when A and C are completed using the topologies induced by the filtration of B. The A-R lemma shows that these topologies are exactly the I-adic topologies on A and C.

Remark: It is easy to see that $\hat{R}$ has a natural ring structure.

Remark: for any R-module M there is a natural map $M \to \hat{M}$ taking m to the equivalence class of the Cauchy sequence with constant value M. The kernel is $\bigcap_n I^n M$.

Remark: For any R-module M

$$[(r_n)], m \mapsto [(r_n m)]$$

is well-defined and gives an R-bilinear map from $\hat{R} \times M$ to $\hat{M}$. This induces a linear map from $\hat{R} \otimes_R M \to \hat{M}$.