CA LECTURE 24

SCRIBE: CLAIRE TOMESCH

Recall: any sequence of modules

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is a chain complex, the homology at A is the kernel of f and the homology at B is the cokernel of f

So a map from one short exact sequence to another gives us an 8-term exact sequence of kernels and cokernels (as explained in the supplement to the notes on lecture 23).

Now we return to studying inverse limits. If A and B are inverse systems of groups then a morphism from A to B is a family of HMs $f = \{f_n : A_n \to B_n\}$ such that

$$\begin{tikzcd}
A_0 \arrow[d, "f_0"] & A_1 \arrow[l, "\pi^A_{10}"'] \arrow[d, "f_1"] & A_2 \arrow[l, "\pi^A_{21}"'] \arrow[d, "f_2"] & \cdots \arrow[l] \\
B_0 & B_1 \arrow[l, "\pi^B_{10}"] & B_2 \arrow[l, "\pi^B_{21}"] & \cdots \arrow[l]
\end{tikzcd}$$

Clearly it induces a map $\lim\limits_{\leftarrow} f$ from $\lim\limits_{\leftarrow} A$ to $\lim\limits_{\leftarrow} B$, and we have made the $\lim\limits_{\leftarrow}$ construction into a functor.

Theorem: if $0 \to A \to B \to C \to 0$ is exact then $0 \to \lim\limits_{\leftarrow} A \to \lim\limits_{\leftarrow} B \to \lim\limits_{\leftarrow} C$ is exact. If moreover the system A is surjective then $0 \to \lim\limits_{\leftarrow} A \to \lim\limits_{\leftarrow} B \to \lim\limits_{\leftarrow} C \to 0$ is exact.

Before the proof a quick digression on direct sums and direct products: let $\{A_i : i \in I\}$ be a family of R-modules. Then the Cartesian product $\prod_i A_i$ is an R-module with the obvious $+$ and scalar multiplication operations, and we call this the *direct product*. The *direct sum* $\oplus_i A_i$ is the submodule consisting of those $f \in \prod_i A_i$ with $\{i : f(i) \neq 0_{A_i}\}$ finite.

Remark: Let $\pi_i : f \mapsto f(i)$ be the obvious projection map. Then $\prod_i A_i$ and the π_i form a product in the categorical sense (that is to say make I into a category whose objects are the elements of I with only identity morphisms, then a product is just a limit for the diagram which maps i to A_i).

Remark: Similarly define $j_i : A_i \to \oplus_i A_i$ so that $j_i(a) : i \mapsto a$, $j_i(a) : j \mapsto 0$ for $j \neq i$. Then $\oplus_i A_i$ and the j_i form a coproduct, that is a colimit for the diagram from the previous remark.

Remark: If $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ are families of R-modules and we have HMs $\{f_i : A \to B_i : i \in I\}$ then we can form a HM $\prod_i f_i$ in the obvious way.

Proof of theorem: define d_A a map from $\prod A_n$ to $\prod A_n$ by $d_A : (a_n) \mapsto (a_n - \pi^A_{n+1n}(a_{n+1}))$. Then the kernel of d_A is the inverse limit of A. Similarly for B and C.

Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod A_n & \xrightarrow{\prod f_n} & \prod B_n & \xrightarrow{\prod_n g_n} & \prod C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\ 0 & \longrightarrow & \prod A_n & \xrightarrow{\prod f_n} & \prod B_n & \xrightarrow{\prod_n g_n} & \prod C_n & \longrightarrow & 0 \end{array}$$

Easily each row is exact. To see it commutes just observe that

$$d_B(\prod f_n)(a_n) = d_B(f_n(a_n)) = (f_n(a_n) - \pi^B_{n+1n} f_{n+1}(a_{n+1})) = (f_n(a_n) - f_n \pi^A_{n+1n}(a_{n+1})) = (\prod f_n) d_A(a_n)$$

where we used the fact that $f = (f_n)$ is a HM of inverse systems to commute f's and π's.

So we get an exact sequence $0 \to ker(d_A) = \lim\limits_{\leftarrow} A \to \lim\limits_{\leftarrow} B \to \lim\limits_{\leftarrow} C \to coker(d_A)$. To finish note that if A is surjective then easily d_A is surjective, and so $coker(d_A) = 0$.

Cultural note: we can view $A \mapsto coker(d_A)$ as a functor which measures the extent to which $\lim\limits_{\leftarrow}$ preserves exact sequences. Such "derived functors" are important when you go further in homological algebra.

NOTE: THE CONVENTION THAT ALL GROUPS ARE ABELIAN IS NOW TEMPORARILY SUSPENDED WHILE WE DISCUSS TOPOLOGICAL GROUPS.

A *topological group* is a set G equipped with a topology and a group structure such that the maps $g \mapsto g^{-1}$ and $(g,h) \mapsto gh$ are continuous.

Key idea: topological groups are very homogeneous as topological spaces.

Recall that an AM of a topological space X is just a permutation π of X which permutes the open sets, or equivalently such that π and π^{-1} are both continuous. We will refer to maps from G to G which have this property as "topological AMs", of course they need not be AMs for the group structure.

In particular if we fix $g \in G$ then each of the maps $h \mapsto h^{-1}$, $h \mapsto gh$, $h \mapsto hg$, $h \mapsto ghg^{-1}$ is a topological AM.

Recall the separation axioms T_0, T_1, T_2 from the "more topology" handout. In general they are distinct but we show that for topological groups they coincide.

T_1 implies T_2: Consider the continuous map $(g,h) \mapsto gh^{-1}$. By the T_1 property $\{e\}$ is closed, so its preimage $\Delta = \{(g,g) : g \in G\}$ is closed in G^2. Now let $a \neq b$ in G so that $(a,b) \notin \Delta$. The complement of Δ is open, hence is a union of rectangles $U \times V$ with U and V open; so we may find $U \ni a$ and $V \ni b$ open sets with $U \times V \cap \Delta = \emptyset$, that is $U \cap V = \emptyset$.

T_0 implies T_1: It will suffice to show that $\{e\}$ is closed. If we can show thus we can apply the topological AM $h \mapsto gh$ to conclude that $\{g\}$ is closed.

So let $g \neq e$ and try to find an open set which contains g but not e. The T_0 axiom provides either this (in which case we are done) or an open set U with $e \in U$ and $g \notin U$. In this case use the fact that $h \mapsto gh^{-1}$ is a topological AM to see that gU^{-1} is open with $g \in gU^{-1}$ and $e \notin gU^{-1}$.