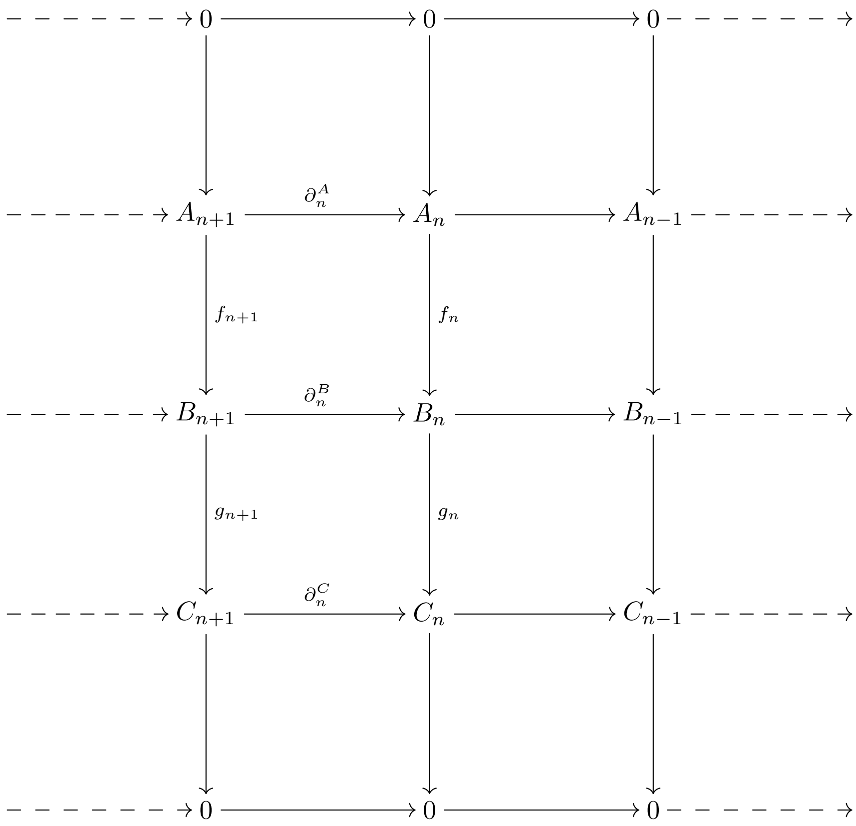
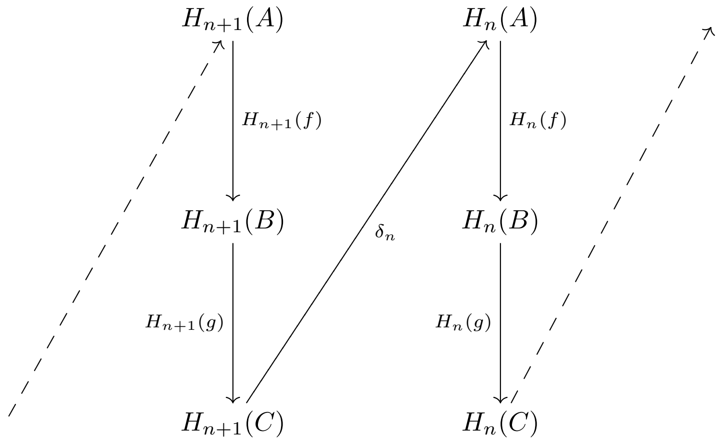
LECTURE 23: THE REST

JC

Now we consider a "short exact sequence" $0 \to A \to B \to C \to 0$ where each of A, B and C are chain complexes. What does this mean? Well we will have a big commutative diagram of R-modules where each row is a chain complex and each column is exact. Commutativity means each square commutes (so that actually along a path from point P to point Q the composition of the morphisms depends only on P and Q).



We will construct an exact sequence (the homology long exact sequence)

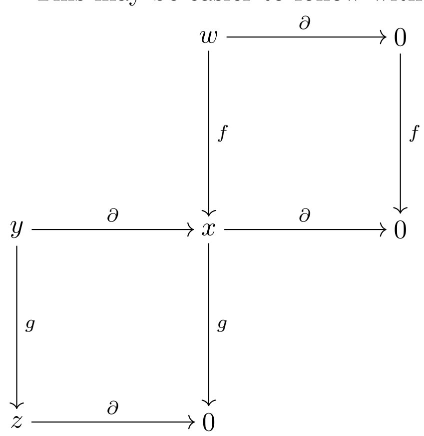


δ_n is called the "connecting HM". We construct it by the method affectionately (?) called "diagram chasing". To simplify notation we drop the subscripts and superscripts from the various maps in the diagram, they should be clear from context.

Let $z \in Z_{n+1}(C)$, so that $\partial z = 0$ (that is to say of course that $\partial_n^C z = 0$). By exactness of column $n+1$ we may choose $y \in B_{n+1}$ such that $gy = z$, where of course the choice of y may not be unique. Let $x = \partial y$. By commutativity $0 = \partial z = \partial gy = g\partial y = gx$, so that by exactness of column n we have $x = fw$ for some $w \in A_n$. Note that by exactness the maps f_j are all injective, in particular w is unique once we know y.

Also by commutativity and the chain complex condition $f\partial w = \partial fw = \partial^2 y = 0$. So by injectivity of f we have $\partial w = 0$ and $w \in Z_n(A)$.

This may be easier to follow with a picture:



We would like to define δ_n to be the map which takes $z + B_{n+1}(C)$ to $w + B_n(A)$, so we need to know that the class of w modulo $B_n(A)$ depends only on the class of z modulo $B_{n+1}(C)$. Since all maps are linear it is enough to show (think about it!) that if $z \in B_{n+1}(C)$ then any choice of w is in $B_n(A)$. So suppose $z = \partial v$ and choose w as above.

Choose u such that $gu = v$. It won't be true in general that $\partial u = y$, but of course we do have $g\partial u = \partial gu = \partial v = z$. So $g(y - \partial u) = 0$, hence by exactness $y - \partial u = ft$ for some t. Finally $f(\partial t - w) = \partial ft - \partial y = 0$ so that $w = \partial t$.

Pictorially we have $$\begin{array}{ccc} u & \xrightarrow{\partial} & \partial u \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ v & \xrightarrow{\partial} & z \end{array}$$ and $$\begin{array}{c} y \\ \downarrow{\scriptstyle g} \\ z \end{array}$$ so we subtract the right hand columns (OK by linearity!) use exactness, commutativity and the chain complex condition to get $$\begin{array}{ccc} t & \xrightarrow{\partial} & \partial t \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ y - \partial u & \xrightarrow{\partial} & x \\ \downarrow{\scriptstyle g} & & \\ 0 & & \end{array}$$ which when compared with $$\begin{array}{c} w \\ \downarrow{\scriptstyle f} \\ x \end{array}$$ gives that $w = \partial t$.

Now we have shown that δ_n is well-defined it is trivial to show it's a HM (remember everything in sight is linear). Of course we still have to check exactness. This is painful but everyone should do it once.

(1) $imH_n(f) = kerH_n(g)$.

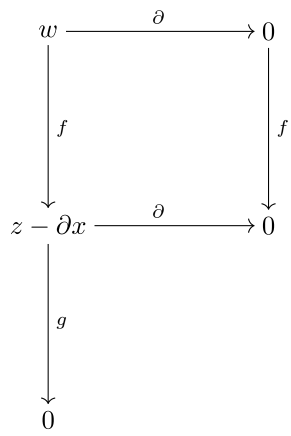
One way is easy: $g \circ f = 0$ and H_n is a functor so $H_n(g) \circ H_n(f) = 0$, that is $imH_n(f) \subseteq kerH_n(g)$.

For the converse let $z \in Z_n(B)$ be such that $z + B_n(B) \in kerH_n(g)$. By definition this mean that $\partial z = 0$ and also that $gz = \partial y$ for some y. Choose x so that $gx = y$, then $g(z - \partial x) = gz - \partial gx = \partial y - \partial y = 0$, so that $z - \partial x = fw$ for some $w \in A_n$. Now $f\partial w = \partial(z - \partial x) = 0$, so $\partial w = 0$ and $w \in Z_n(A)$. We showed that $z + B_n(B) = fw + B_n(B) = H_n(w + B_n(A))$, so that $z + B_n(B) \in \mathrm{im}H_n(f)$.

In pictures:

$$\begin{array}{ccccc} x & & z & \xrightarrow{\partial} & 0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\ y & \xrightarrow{\partial} & \partial y = gz & & \end{array}$$

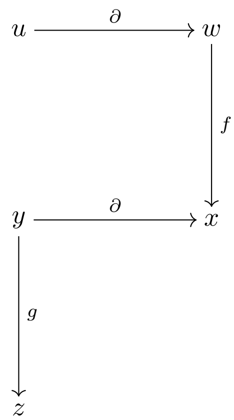
and



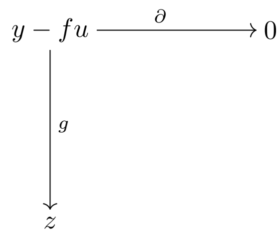
(2) $imH_{n+1}(g) = ker\delta_n$.

Let $z + B_n(C)$ be in the kernel of δ_n. So there exist $u \in A_{n+1}$ and $y \in B_{n+1}$ such that $gy = z$ and $\partial y = f\partial u$. Now $\partial(y - fu) = \partial y - f\partial u = 0$, that is $y - fu \in Z_{n+1}(B)$. By exactness $g(y - fu) = gy = z$, so that $H_{n+1}(g) : (y - fu) + Z_{n+1}(B) \mapsto z + Z_{n+1}(C)$.

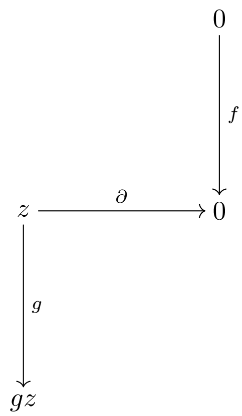
Pictures:



and

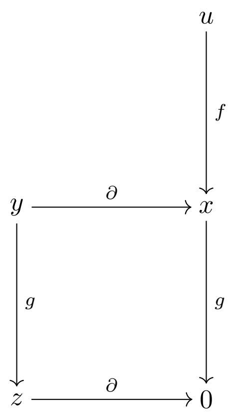


Now let $z \in Z_{n+1}(B)$ and consider the construction of $\delta_n(gz+B_{n+1}(C))$. Since $\partial z = 0$ it is easy to see that $\delta_n : gz + B_{n+1}(C) \mapsto 0 + B_n(A)$.

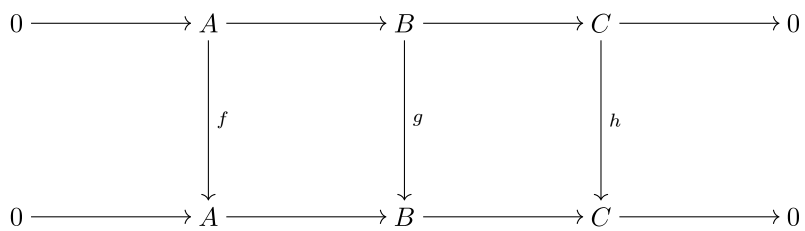


(3) $im\delta_n = kerH_n(f)$.

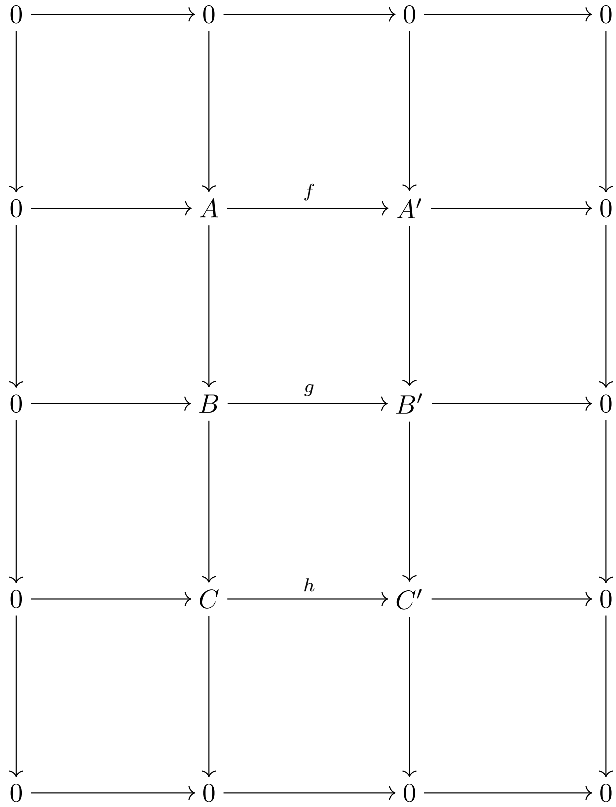
This one is actually easy! For $u \in Z_n(A)$ we see that $u + B_n(A)$ is in $ker H_n(f)$ iff $fu \in B_n(B)$ iff there is $y \in B_{n+1}$ such that $\partial y = fu$ iff there is $z \in Z_{n+1}(C)$ such that $\delta_n : z + B_{n+1}(C) \mapsto u + B_n(A)$.



For use on Friday we isolate a special case. Suppose we have a commutative diagram with exact rows



Padding it with zeroes and rotating it we get a diagram where the rows are chain complexes and the columns are short exact sequences



In this case the homology long exact sequence gives us

$$0 \to ker(f) \to ker(g) \to ker(h) \to coker(f) \to coker(g) \to coker(h) \to 0$$