CA LECTURE 21

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Defn: a DVR is an ID R such for some valuation v on the field of fractions of R, R equals the valuation ring.

We saw last time that if R is a DVR and $t \in R$ has valuation 1 then the ideals $I \neq 0, R$ are of the form (t^n) for $n \geq 1$. In particular R is a local N'ian ID of dimension one (last part is easy because (t) will be the only nonzero prime ideal).

We will give some equivalent characterisations: but first some general facts about local N'ian IDs of dimension one.

Let R be a local N'ian ID of dimension one, with unique nonzero prime ideal P. Then we claim that 1) Every ideal $I \neq 0, R$ is P-primary and also $P^n \subseteq I$ for some n 2) $P^n \neq P^{n+1}$ for all n.

Proof: 1) Since R is N'ian I is the intersection of nonzero primary ideals, which must all be P-primary since P is the only nonzero prime ideal. An intersection of P-primary ideals is P-primary so I is P-primary. Finally R/I is Noetherian so the nilradical of R/I (which is P/I) is nilpotent, hence $P^n \subseteq I$ for some n. 2) By our general analysis of N'ian local rings, if $P^n = P^{n+1}$ for some n then R is Artinian, but then R would have dim zero.

Now for the main theorem:

Theorem 1. *Let R be a local N'ian ID of dimension 1. Let M be the unqique nonzero prime ideal, $k = R/M$, K the field of fractions. Then TFAE*

(1) *R is a DVR.*
(2) *R is integrally closed.*
(3) *M is principal.*
(4) *The k-dimension of M/M^2 is one.*
(5) *Every ideal $I \neq 0, R$ is of form M^n.*
(6) *There is t such that every ideal $I \neq 0, R$ is of form (t^n).*

Proof. 1 implies 2: Let v be a discrete valn on K such that R is the valn ring. Let $a \in K$ be integral over R, say $a^n = \sum_{i<n} r_i a^i$. If $a \in R$ we are done. Otherwise $v(a) < 0$ so $v(a^{-1}) = -v(a) > 0$, and $a^{-1} \in R$. But now multiplying by a^{-n-1} we have $a = \sum_{i<n} r_i a^{i-n-1}$, so $a \in R$ after all.

2 implies 3: Let a be any nonzero nonunit (they must exist since R has dimension one, so is not a field). The ideal $(a) \neq 0, R$ so is M-primary and contains M^n for some n. Let n be minimal such that $M^n \subseteq (a)$, so that $M^{n-1} \nsubseteq (a)$ (by convention $M^0 = R$). Choose $b \in M^{n-1} \setminus (a)$ and set $x = b/a$, noting that since $b \notin (a)$ we have $x \notin R$.

By integral closure x is not integral over R. Now by N'ian hypothesis M is fg as an R-module; were it the case that $xM \subseteq M$ then M would be a faithful $R[x]$-module, but this is impossible since it would imply x integral over R. So $xM \nsubseteq M$. On the other hand $b \in M^{n-1}$ so $bM \subseteq M^n \subseteq (a)$, hence $xM = (bM)/a \subseteq R$ and so xM is an ideal of R not contained in M

Now xM contains a unit as R is local with unique maxl ideal M. So $xM = R$, and so easily $x^{-1} \in M$ and $M = x^{-1}R$ is principal.

3 implies 4. Since M is principal, M/M^2 is spanned by a set of size one so the k-dimension of M/M^2 is at most one. But $M \neq M^2$ so M/M^2 is not the zero space, hence the k-dimension is exactly one.

4 implies 5: Let $I \neq 0, R$ so that as usual $I \supseteq M^n$. Consider the quotient ring $R' = R/M^n$. This is clearly local with maxl ideal $M' = M/M^n$. It is easy to see that $k' = R'/M'$ is isomorphic to $k = R/M$, and also that M/M^2 is IMic to $M'/(M')^2$. We also know by an old remark that R/M^n is an Artin local ring, so by our analysis of such rings every ideal of R' is a power of M'. So I/M^n is a power of M', from which it follows that I is a power of M.

5 implies 6: Choose $t \in M \setminus M^2$. Then $(t) \subseteq M$, $(t) \nsubseteq M^2$ so the only possibility is $(t) = M$. By hypothesis every $I \neq 0, R$ has form $M^n = (t)^n = (t^n)$.

6 implies 1: Let $u \in R$ be a nonzero nonunit and let $v(u) = n$ for the unique $n \geq 1$ with $(u) = (t^n)$. Defibe $v(0) = \infty$, $v(u) = 0$ for all units u. Extend to K in the obvious way and check this is a discrete valuation with valuation ring R.

□

Pregnant remark: if v is the p-adic valn on $\mathbb{Q}$ then the valuation ring is just the localisation $\mathbb{Z}_{(p)}$.

Theorem 2. *Let R be a N'ian ID of dimension one. Then every ideal $I \neq 0, R$ can be written uniquely as a product of nonzero primary ideals with distinct radicals.*

Proof. Existence: since I is N'ian, we have an irredundant primary decomposition $I = Q_1 \cap \ldots Q_m$. If Q_i is P_i -primary then the P_i are distinct maximal ideals, so easily they are pairwise comaxl. Hence the Q_i are pairwise comaxl so by the CRT $I = Q_1 \ldots Q_m$.

Uniqueness: Suppose $I = Q_1 \ldots Q_m$ where Q_i is P_i-primary and the P_i are distinct. Then arguing as above the Q_i are pairwise comaxl so $I = Q_1 \cap \ldots Q_m$ and we have an irredundant primary decomposition. Since the P_i maximal we never have $P_i \subsetneq P_j$ so that there are no embedded primes in this primary decomposition. Hence by the uniqueness theorems on decompositions the Q_i are unique. □