CA LECTURE 18

SCRIBE: YIMU YIN

An R-module is *Artinian* iff it satisfies one of the following equivalent conditions:

(1) Every decreasing sequence of modules $M_0 \geq M_1 \ldots$ stabilises.
(2) Every nonempty family of submodules has a minimal element.

The ring R is *Artinian* iff it is Artinian as an R-module.

Some examples (all to be understood as $\mathbb{Z}$-modules)

(1) $\mathbb{Z}$ is N'ian and not A'ian.
(2) The subgroup of $\mathbb{Q}/\mathbb{Z}$ whose elements have order a power of two is A'ian but not N'ian.
(3) The subgroup of $\mathbb{Q}$ consisting of elements with denominator a power of two is neither.

The elementary theory of A'ian modules is quite like the theory for N'ian modules.

Theorem: let $M \leq N$ be R-modules. Then TFAE

(1) N is A'ian
(2) Both M and N/M are A'ian.

The forward direction is easy. So suppose that M and N/M are A'ian and that $N_0 \geq N_1 \ldots$ is a decreasing chain of submodules of N.

Choose i so large that $N_i \cap M = N_j \cap M$ and $(N_i + M)/M = (N_j + M)/M$ for $j \geq i$. Let $n \in N_i$ so that $n + M \in (N_i + M)/M = (N_j + M)/M$, and we find $\bar{n} \in N_j$ so that $n + M = \bar{n} + M$. Then $n - \bar{n} \in N_i \cap M = N_j \cap M$, so that $n \in N_j$. So $N_i = N_j$ and we are done.

Now (just like we did for N'ian modules) routine to argue that if R is an Artinian ring then R^n is an A'ian R-module for all n and thus every fg R-module is A'ian.

Now we derive some more information about N'ian rings.

Defn: an ideal I is *irreducible* iff whenever $I = J \cap K$ for ideals J and K then $I = J$ or $I = K$.

Theorem: in a Noetherian ring every ideal is a finite intersection of irreducible ideals.

Proof: let I be a maximal counterexample (must exist since R is N'ian). Then I is not irred so $I = J \cap K$ with $I \neq J, K$. But then $I \subsetneq J, K$ so that J, K are finite intersections of irreducible ideals. So I is a finite interesction of irreducible ideals, contradiction!

Theorem: in a N'ian ring if $I \neq R$ is an irreducible ideal then it is primary.

Proof: going to R/I it is enough to show that in a N'ian ring, if (0) is irreducible then its primary. So let $xy = 0$ with $y \neq 0$. let I_n be the ideal of z such that $zx^n = 0$, then the chain of I's increases so must stabilise. Let $I_n = I_{n+1}$, then we claim that $(y) \cap (x^n) = (0)$; for if bx^n is a multiple of y then $bx^n x = 0$, so that $b \in I_{n+1} = I_n$ and so $bx^n = 0$. Since (0) is irreducible, $(x^n) = 0$ and so $x^n = 0$.

Theorem: in an Artinian ring every prime ideal is maximal.

Proof: ETS that an Artinian ID is a field. Let $x \neq 0$ and find n so that $(x^n) = (x^{n+1})$ using the A'ian hypothesis on $(x) \supseteq (x^2) \ldots$. Then $x^n = yx^{n+1}$ for some y and so by cancellation $xy = 1$.

Theorem: An A'ian ring has finitely many maximal ideals.

Proof. WLOG R is nonzero. Let I be minimal among the ideals which are finite interesctions of maximal ideals and let $I = M_1 \cap \ldots M_k$. Then for any maximal M we have $M \cap I = I$ so $I = M_1 \cap \ldots M_k \subseteq M$. As the M_i are prime we have $M_i \subseteq M$ and hence $M_i = M$.

General discussion: let R be a ring with a maxl ideal I and an ideal J. We can view J and IJ as R-submodules of R. The quotient J/IJ can be viewed as an R-module or as an R/I-module, that is to say as a vector space over R/I. Also the ideals K with $IJ \subseteq K \subseteq J$ are in bijection with the R-submodules of J/IJ, and these are precisely the subspsces in the R/I-vector space structure on J/IJ.