CA LECTURE 17

SCRIBE: NO SCRIBE!

Start with some easy results that we use a bit later.

Let M be an R-module. Let K and L be R-modules and let $S \subseteq R$ be MC. Then

(1) $S^{-1}R = 0$ iff $0 \in S$.

Proof: If $0 \in S$ then everything gets identified. Conversely if $S^{-1}R = 0$ then $0/1 = 1/1$, so by definition there is $u \in S$ such that $u(1) = 0$.

(2) $S^{-1}K \cap L = S^{-1}K \cap S^{-1}L$.

Proof: Obviously $S^{-1}K \cap L \subseteq S^{-1}K \cap S^{-1}L$. Conversely let $k/s = l/t$ for $k \in K$, $l \in L$ and $s, t \in S$. By definition there is $u \in S$ such that $u(tk - sl) = 0$, that is $tuk = sul \in K \cap L$. So $k/s = tuk/stu = sul/stu = l/t$ and we are done.

Now let A and B be rings with $A \leq B$ and let $S \subseteq A$ be MC. Then S is MC in B and we may form a ring $S^{-1}B$. It is easy to see that $S^{-1}A \leq S^{-1}B$ either directly or by applying S^{-1} to the exact sequence of A-modules $0 \to A \to B$. Let C be the integral closure of A in B. Then we claim that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

Firstly let $c \in C$ and $s \in S$. Then c is integral over A, say $c^n = \sum_{i<n} a_i c^i$ for some $a_i \in A$. Working in $S^{-1}B$ we may divide through by the unit s^n to get an equation $c^n/s^n = \sum_{i<n} a_i/s^n c^i = \sum a_i/s^{n-i}(c/s)^i$, so that c/s is integral over $S^{-1}A$. More elegant proof: take a ring which contains A and c and is module finite/A, apply S^{-1} and observe that module-finiteness is preserved.

Now suppose that b/t is integral over $S^{-1}A$. Since t is a unit in $S^{-1}A$ it is easy to see that $b/1$ is integral over $S^{-1}A$, that is

$$b^n/1 = \sum_{i<n} a_i/s_i b^i/1$$

Letting $s = \prod_i s_i$ and $s_i^* = \prod_{j \neq i} s_j$ we get

$$b^n/1 = \sum_{i<n} a_i s_i^* b^i/s,$$

so that

$$ub^n s = \sum_{i<n} u a_i s_i^* b^i$$

for some $u \in S$. Finally multiplying by $u^{n-1}s^{n-1}$

$$u^n b^n s^n = \sum_{i<n} a_i s_i^* u^{n-i} s^{n-1-i} (ubs)^i,$$

so that $ubs \in C$ and we are done.

Corollaries: let $A \leq B$ and let $S \subseteq A$ be mc. Then if B is integral over A, $S^{-1}B$ is integral over $S^{-1}A$. Similarly if A is integrally closed in B, $S^{-1}A$ is integrally closed in $S^{-1}B$.

We also have the following easy remark: if $A \leq B$, J is an ideal of I and $B \cap J = I$ then B/J is integral over A/I. Note: for this to make sense we need to identify A/I with a subring of B/J via the injective HM $a + I \mapsto a + J$.

Integral closures: by a recent HW it makes so sense to define "integral closure" by analogy with "algebraic closure" (because we can always add new integral elements in some ring extension). So we make the following definition for R an ID: R is *integrally closed* iff it is integrally closed in its field of fractions.

Fact: if R is a UFD it is integrally closed.

Proof: We may write any element of the FOF a/b so that a and b have no prime factors in common. If a/b is integral over R then $(a/b)^n = \sum_{i<n} r_i a^i/b^i$, so that $a^n = \sum_{i<n} r_i a^i b^{n-i}$. So any prime divisor of b divides a, hence b has no prime divisors and thus b is a unit and $a/b \in R$.

Now we study the relation between primes of A and B when B is integral over A.

Fact: let A and B be IDs with B integral over A. Then B is a field iff A is a field.

Proof: First let A be a field and let $b \neq 0$ be in B. Let $b^n + \sum_{i<n} a_i b^i = 0$ with $a_i \in A$ and n minimal. Since B is an ID and $b \neq 0$ we must have $a_0 \neq 0$, for otherwise $b^{n-1} + \sum_{0<i<n} a_i b^{i-1} = 0$ contradicting mininmal choice of n. Now it is easy to check that $b^{-1} = -a_0^{-1}(b^{n-1} + \sum_{0<i<n} a_i b^{i-1})$.

Now let B be field and let $a \neq 0$ be in A. Then $a^{-1} \in B$ so is integral over A that $a^{-n} = \sum_{i<n} a_i a^{-i}$. Mutiplying by a^{n-1} we see $a^{-1} = \sum_{i<n} a_i a^{n-1-i} \in A$.

Fact: let $A \leq B$ be rings with B integral over A. Let P and Q be prime ideals of A and B respectively with $Q \cap A = B$. Then P is maxl in A iff Q is maxl in B.

Proof: observe that $A/P \leq B/Q$, B/Q is integral over A/P, and both rings are IDs. Then use last fact.

Theorem: let $A \leq B$ with B integral over A. Then for every prime ideal P of A there is a prime ideal Q of B with $Q \cap A = P$.

Proof: Let $S = A \setminus P$ and form $S^{-1}A$ and $S^{-1}B$ as usual (this latter object is the ring which is confusingly called B_P in A and M's text). Since B is integral over A, $S^{-1}B$ is integral over $S^{-1}A$.

As we pointed out above $S^{-1}B \neq 0$ and so it has a maximal ideal M say. By the standard result on prime ideals in a ring of fractions, $M = S^{-1}Q$ for some prime ideal Q with $Q \cap S = \emptyset$. As usual $Q \cap A$ is prime in P, and we have $Q \cap A \subseteq P$ because Q avoids S.

Now we also have (thinking of A, B, Q as A-modules) that $S^{-1}(Q \cap A) = M \cap S^{-1}A$. Since $S^{-1}B$ is integral over $S^{-1}A$, $M \cap S^{-1}A$ is maximal in $S^{-1}A$. So it is the unique maximal ideal of this local ring, that is $S^{-1}P = S^{-1}(Q \cap A)$. We know that the map $I \mapsto S^{-1}I$ sets up a bijection between primes of R avoiding S (that is contained in P) and primes of $S^{-1}R$. So we get that $Q \cap A = P$ as required.

Theorem (going up): Let $A \leq B$ with B integral over A. Let $0 < m < n$ and suppose we are given P_i (for $1 \leq 1 \leq n$) and Q_j (for $1 \leq j \leq m$) so that the P's are an increasing chain of prime ideals in A, the Q's are an increasing chain of prime ideals in B, and $Q_j \cap A = P_j$ for $j \leq m$. Then we can prolong the chain of Q's to a chain of length n with the same properties.

Proof: it is enough to do $m = 1 < n = 2$. Note that $A/P_1 \leq B/Q_1$ and B/Q_1 is integral over A/P_1. Now P_2 corresponds to a prime ideal (mild abuse of notation here!) P_2/P_1 in A/P_1 (it is prime because the quotient of A/P_1 by P_2/P_1

is isomorphic as a ring to A/P_2). So by the last results we get a prime ideal $\bar{Q}$ with $\bar{Q} \cap A/P_1 = P_2/P_1$. Now $\bar{Q} = Q_2/Q_1$ for some ideal Q_2 of B/Q_1, and it is easy to check that Q_2 is prime and $Q_2 \cap A = P_2$.