CA LECTURE 14

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Fix an R-module N. We show how to make the operation "tensor with N" into a covariant functor from R-modules to R-modules. This is actually easy: given an R-linear map $f : M_1 \to M_2$ we observe that $(m, n) \mapsto f(m) \otimes n$ is an R-bilinear map from $M_1 \times N$ to $M_2 \otimes N$, so there is a unique linear map from $M_1 \otimes N$ to $M_2 \otimes N$ such that $m \otimes n \mapsto f(m) \otimes N$. In the notation of an earlier lecture this is $f \otimes id_N$. It is routine to check we have defined a functor and also that this functor maps zero modules to zero modules and zero maps to zero maps.

We already saw (in the example where G is a group of order two and the inclusion map $\mathbb{Z} \to \mathbb{Q}$ induces a zero map $\mathbb{Z} \otimes G \to \mathbb{Q} \otimes G$) that tensoring does not preserve exactness in general. We show it does preserve some exactness.

Consider R-modules A, B, C. It is routine to check that $Hom(A, Hom(B, C))$ is isomorphic as an R-module to the space of bilinear maps from $A \times B$ to C, via the map which takes ϕ to $(a, b) \mapsto \phi(a)(b)$. We also have an isomorphism of R-modules between the set of bilinear maps from $A \times B$ to C and $Hom(A \otimes B, C)$. So we get an IM between $Hom(A, Hom(B, C))$ and $Hom(A \otimes B, C)$.

To be explicit if ψ is a linear map from $A \otimes B$ to C then the corresponding map in $Hom(A, Hom(B, C))$ is $a \mapsto (b \mapsto \psi(a \otimes b))$.

Exercise for a rainy night: convince youself that the IM we just gave is natural in A, B, C and in particular sets up for each B an adjunction between $- \otimes B$ and $Hom(B, -)$.

Theorem 1. *Let $M_1 \to M_2 \to M_3 \to 0$ be exact. Then for all N the sequence $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ obtained by tensoring with N is exact.*

Proof. Applying the forward direction(s) of a result from last time, for all N and P the sequence

$$0 \to Hom(N, Hom(M_3, P)) \to Hom(N, Hom(M_2, P)) \to Hom(N, Hom(M_1, P))$$

obtained by applying $Hom(N, -)$ and $Hom(-, P)$ is exact. Applying the IM we discussed above

$$0 \to Hom(N \otimes M_3, P)) \to Hom(N \otimes M_2, P)) \to Hom(N \otimes M_1, P))$$

is exact for all P. Finally applying the reverse direction of the result from last time

$$M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

is exact.

Notes: I used the IM between $M \otimes N$ and $N \otimes M$. Also one should check that the arrows between the various modules are right; this follows from the "naturalness" part of the adjunction between $- \otimes N$ and $Hom(N, -)$. ∎

New topic: decomposing ideals.

Start with some easy general facts about radicals and prime ideals.:

(1) $\sqrt{J} = \{a : \exists n > 0 : a^n \in J\}$. In the correspondence between ideals of R and ideals of R/J this answers to the nilradical of R/J.
(2) J is *radical* iff $\sqrt{J} = J$. $\sqrt{\sqrt{I}} = \sqrt{I}$ so $\sqrt{I}$ is radical. Prime ideals are radical.
(3) We have $\sqrt{I_1 \cap \ldots I_n} = \sqrt{I_1} \cap \ldots \sqrt{I_n}$. One inclusion is easy, the other holds because if $x \in \sqrt{I_i}$ then all sufficiently high powers of x are in I_i.
(4) If P is prime and $I_1 \cap \ldots \cap I_n \subseteq P$ then for some j we have $I_j \subseteq P$. Proof: if not we choose $a_j \in I_j \setminus P$, let a be the product and note that $a \in I_1 \cap \ldots I_n \setminus P$. In particular if $P = I_1 \cap \ldots I_n$ then $P = I_j$ for some j.
(5) Recall IJ is the ideal generated by all products ab for $a \in I$, $b \in J$. I^n is generated by all products $a_1 \ldots a_n$ with $a_i \in I$ (not just by a^n for $a \in I$!) Note that $I^n \subseteq I$.
    For any ideal I and any $n > 0$ we have $I \subseteq \sqrt{I^n} \subseteq \sqrt{I}$, and if I is radical then $I = \sqrt{I^n}$.

Definition 1. *An ideal Q is* primary *iff $Q \neq R$ and for all a and b, $ab \in Q$ implies $a \in Q$ or $b \in \sqrt{Q}$.*

Equivalent: $R/Q \neq 0$ and every zero-divisor in R/Q is nilpotent.

Easy to see: if Q primary then $\sqrt{Q}$ is prime. We say Q is *P-primary* if it is primary with $\sqrt{Q} = P$.

In general $\sqrt{Q}$ prime does not imply Q radical.

Fact: If $\sqrt{Q}$ is *maximal* then Q is primary. To see this note that in R/Q the nilradical is maximal, then appeal to an old HW to see that all elements of R/Q are units or nilpotent.

Fact: the intersection of finitely many P-primary ideals is P-primary. Routine.

We say that I is *decomposable* iff it is a finite intersection of primary ideals.

Cosmetics: Suppose I is decomposable. The deocomposition $I = Q_1 \cap \ldots Q_n$ is *irredundant* iff the radicals of the Q_i are distinct and $Q_i \not\supseteq \cap_{j \neq i} Q_j$ for all i. Given any decomposition of I we may group the primary ideals by radical, intersect each group and discard the junk to get an irredundant decomposition; or if you prefer any decomposition with a minimal number of primary ideals is automatically irredundant.