ROUGH VERSION OF SEP 28 LECTURE NOTES

JC

Left over from last time: let R be a ring and I an ideal of R, S a MC subset. We have an exact sequence of R-modules

$$0 \to I \to R \to R/I \to 0$$

NB: It is NOT an exact sequence of rings, some of the objects have no ring structure and the arrow $0 \to R$ is not a ring HM.

So applying S^{-1} we have an exact sequence of $S^{-1}R$-modules

$$0 \to S^{-1}I \to S^{-1}R \to S^{-1}R/I \to 0$$

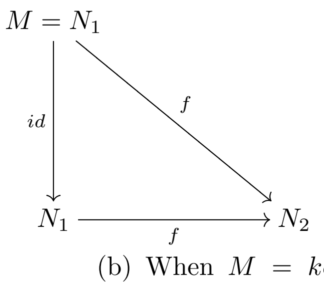
So $S^{-1}I \leq S^{-1}R$ and $S^{-1}R/I \simeq S^{-1}R/S^{-1}I$ as $S^{-1}R$-modules. This resembles the conclusion of an old HW but is formally different: you should convince yourself that there is a bijection between the module of fractions $S^{-1}R/I$ and the ring of fractions $\bar{S}^{-1}R/I$, and that the induced bijection between $\bar{S}^{-1}R/I$ and $S^{-1}R/S^{-1}I$ is an IM of rings.

Now we study the Hom-sets of the category of modules. Start by noting that if R is a ring and M and N are R-modules then $Hom(M, N)$ (which is the set of R-linear maps from M to N) has a natural R-module structure with operations of pointwise addition and scalar multiplication. What is more if $f : N_1 \to N_2$ then the map $Hom(M, f) : \gamma \mapsto f \circ \gamma$ is easily seen to be an R-linear map from $Hom(M, N_1)$ to $Hom(M, N_2)$. It is routine to check we defined a covariant functor $Hom(M, -)$ from R-modules to R-modules.

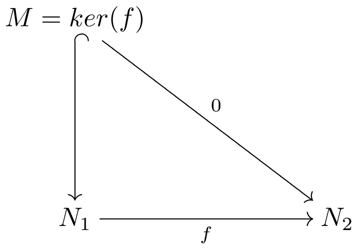
By similar considerations we can define a functor $Hom(-, N)$ where for $g : M_1 \to M_2$ we have $Hom(g, N) : \delta \mapsto \delta \circ g$ from $Hom(M_2, N)$ to $Hom(M_1, N)$. This is contravariant. We may of course also think of $Hom(-, -)$ as a functor from $R - Mod^{op} \times R - Mod$ to $R - Mod$.

Unlike S^{-1} these functors do not preserve all exact sequences. Pursuing the question of just how much exactness they preserve would (and eventually will) lead us in the direction of *homological algebra*. For now we content ourselves with some easy positive results. Before we state and prove them a few easy remarks:

(1) Let $f : N_1 \to N_2$.
   (a) When $M = N_1$ and $\gamma : M \to N_1$ is id_M then $Hom(M, f)(\gamma) = f$.



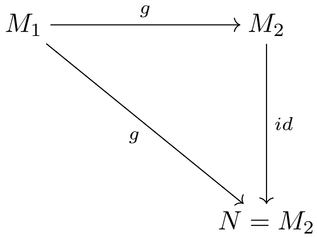
   (b) When $M = ker(f)$ and $\gamma : M \to N_1$ is the inclusion map then $Hom(M, f)(\gamma) = 0$.



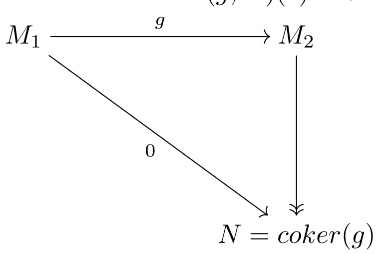
Cultural note: It is not hard to see that the the inclusion HM from $ker(f)$ to N_1 is a final object in a category whose objects are maps γ into N_1 such that $f \circ \gamma = 0$.

(2) Let $g : M_1 \to M_2$.

(a) When $N = M_2$ and $\delta : M_2 \to N$ is id then $Hom(g, N)(\delta) = g$.



(b) The *cokernel* of a module HM $\rho : A \to B$ is $coker(\rho) = B/im(\rho)$, where we note that ρ is surjective iff it has the zero module as cokernel. When $N = coker(g)$ and $\delta : M_2 \to N$ is the quotient map then $Hom(g, N)(\delta) = 0$.



Cultural note: It is not hard to see that the the quotient HM from M_2 to $coker(g)$ is an initial object in a category whose objects are maps δ from M_2 such that $\delta \circ g = 0$.

Note the "duality" between the first set if claims and the second.

Now we state an essentially trivial but useful fact about exact sequences.

Theorem 1. (1) *Let* $0 \to M_1 \to M_2 \to M_3$ *be a (not necessarily exact) sequence of* R*-modules and HMs. Then the following are equivalent:*

(a) $0 \to M_1 \to M_2 \to M_3$ *is exact.*

(b) *For all* N *the corresponding sequence* $0 \to Hom(N, M_1) \to Hom(N, M_2) \to Hom(N, M_3)$ *is exact.*

(2) *Let* $M_1 \to M_2 \to M_3 \to 0$ *be a (not necessarily exact) sequence of* R*-modules and HMs. Then the following are equivalent:*

(a) $M_1 \to M_2 \to M_3 \to 0$ *is exact.*

(b) *For all* N *the corresponding sequence* $0 \to Hom(M_3, N) \to Hom(M_2, N) \to Hom(M_1, N)$ *is exact.*

Proof. We prove the second part. A proof of the first can be obtained by "reversing all the arrows".

So suppose first that $M_1 \to M_2 \to M_3 \to 0$ is exact where $\alpha : M_1 \to M_2$ and $\beta : M_2 \to M_3$, and let N be arbitrary. Exactness amonts to saying that β is surjective and $im(\alpha) = ker(\beta)$.

If $\gamma : M_3 \to N$ and $Hom(\beta, N)(\gamma) = \gamma \circ \beta = 0$, then since β is surjective it follows that $\gamma = 0$. So $Hom(\beta, N)$ has trivial kernel, and the sequence $0 \to Hom(M_3, N) \to Hom(M_2, N) \to Hom(M_1, N)$ is exact at $Hom(M_3, N)$.

We know $\beta \circ \alpha = 0$ so $Hom(\alpha, N) \circ Hom(\beta, N) = 0$, and this gives us one direction (image contained in kernel) of exactness at $Hom(M_2, N)$. For the other direction let $\gamma : M_2 \to N$ be such that $Hom(\alpha, N)(\gamma) = \gamma \circ \alpha = 0$. So γ is zero on $im(\alpha) = ker(\beta)$ and so (check it) we may define $\delta : M_3 \to N$ so that $\delta(\beta(x)) = \gamma(x)$ and thus $\gamma = \delta \circ \beta = Hom(\beta, N)(\delta)$.

Now suppose that $0 \to Hom(M_3, N) \to Hom(M_2, N) \to Hom(M_1, N)$ is exact for all N. To show that β is surjective let N be the cokernel of β and let γ be the quotient map. Then $\gamma \circ \beta = 0$ so by the injectivity of $Hom(\beta, N)$ we get $\gamma = 0$, that is $im(\beta) = M_3$. So we have verified exactness at M_3.

Now let $N = M_3$ and $\gamma = id$. Then we see that β is in the image of $Hom(\beta, N)$ and this in the kernel of $Hom(\alpha, N)$, that is $\beta \circ \alpha = 0$ and so $im(\alpha) \subseteq ker(\beta)$. Finally let N be the cokernel of α and $\gamma : M_2 \to N$ the projection map, so γ is in the kernel of $Hom(\alpha, N)$ and so by exactness is in the image of $Hom(\beta, N)$, that is to say that for some $\delta : M_3 \to N$ we have $\gamma = \delta \circ \beta$. But now for all $x \in ker(\beta)$ we have $\gamma(x) = 0$ and hence $x \in ker(\alpha)$.

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