CA LECTURE 12

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Let R be a ring, M an R-module and $S \subseteq R$ an MC set. We show that $S^{-1}M$ and $S^{-1}R \otimes_R M$ are IMic as $S^{-1}R$-modules.

Consider the map $(a/s, m) \mapsto am/s$ from $S^{-1}R \times M$ to $S^{-1}M$. Routinely it is well-defined and R-bilinear, we show it is initial in the category of R-bilinear maps on $S^{-1}R \times M$.

So let $\phi : S^{-1}R \times M \to N$ be bilinear. We claim that if $am/s = bn/t$ then $\phi(a/s, m) = \phi(b/t, n)$. By defn if $am/s = bn/t$ there is $u \in S$ such that $u(atm - bsn) = 0$. Now we have

$$\phi(a/s, m) = \phi(atu/stu, m) = \phi(1/stu, atum) =$$

$$\phi(1/stu, bsun) = \phi(bsu/stu, n) = \phi(b/t, n),$$

by a series of appeals to the R-linearity in each variable. Now attempt to define $\gamma : S^{-1}M \to N$ by $\gamma : m/s \mapsto \phi(1/s, m)$. By the calculation above γ is well-defined. R-linearity follows from the calculations

$$\gamma(am/s) = \phi(1/s, am) = a\phi(1/s, m) = a\gamma(1/s, m),$$

$$\gamma(m/s + n/t) = \gamma((tm + sn)/st) = \phi(1/st, tm + sn) =$$

$$= \phi(1/st, tm) + \phi(1/st, sn) = \phi(1/s, m) + \phi(1/t, n) = \gamma(m/s) + \gamma(n/t).$$

So by uniqueness of initial objects there is a unique IM of R-modules β from $S^{-1}R \otimes_R M$ to $S^{-1}M$ with $\beta : a/s \otimes m \mapsto am/s$. Notice that since this is an IM every element of the tensor product can be written as $1/t \otimes m$ for some $m \in M$ and $t \in S$ (we could have proved this directly, see A and M Prop 3.5)

Jonathan Gross makes the reasonable complaint that we promised an IM of $S^{-1}R$-modules so we verify that β behaves well wrt the scalar multiplication by elements of $S^{-1}R$. This is easy:

$$\beta(a/s(1/t \otimes m) = \beta(a/st \otimes m) = am/st = a/s\beta(1/t \otimes m).$$

Fact: let R be a ring, M and N be R-modules and $S \subseteq R$ is MC. Then $S^{-1}(M \otimes_R N) \simeq S^{-1}M \otimes_{S^{-1}R} S^{-1}N$ as $S^{-1}R$-modules.

We use the IM thm from the end of last time and the observation that the HM $a \mapsto a/1$ from R to $S^{-1}R$ makes $S^{-1}R$ into an R-algebra. In particular every $S^{-1}R$-module (including $S^{-1}R$!) is naturally an $(R, S^{-1}R)$-bimodule and vice versa. All the isomorphisms in the proof that follows are to be construed as IMs of $(R, S^{-1}R)$-bimodules.

We know that $S^{-1}M \otimes_{S^{-1}R} S^{-1}N \simeq (M \otimes_R S^{-1}R) \otimes_{S^{-1}R} (S^{-1}R \otimes_R N)$ By two appeals to the associative law from last time $(M \otimes_R S^{-1}R) \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \simeq M \otimes_R (S^{-1}R \otimes_{S^{-1}R} S^{-1}R) \otimes_R N$. For any ring B we have $B \otimes_B B \simeq B$, so $M \otimes_R (S^{-1}R \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \simeq M \otimes_R S^{-1}R \otimes_R N$, and finally $M \otimes_R S^{-1}R \otimes_R N \simeq S^{-1}R \otimes_R (M \otimes_R N) \simeq S^{-1}(M \otimes_R N)$. A little thought shows that under the IM we have $m/s \otimes n/t \mapsto 1/st(m \otimes n)$.

We just saw that the S^{-1} construction commutes with the $\otimes$ construction. It can be observed that actually S^{-1} commutes with many constructions in algebra. As a partial explanation of this fact we define "exact sequences" and show that S^{-1} preserves exactness.

Fix a ring R. An *exact sequence* of R-modules is a sequence of maps $f_i : M_{i-1} \to M_i$ such that $im(f_i) = ker(f_{i+1})$.

We let 0 denote the zero module, and note that this is both initial and final in the category of R-modules. A *short exact sequence* is a sequence of the form $0 \to M \to N \to P \to 0$. Now the map $M \to N$ is injeective, the map $N \to P$ is surjective, and by the first IM theorem $P \simeq N/M'$ where M' is the isomorphic copy of M given by the image of the injective map $M \to N$.

Cultural note: we can break down a long exact sequence of maps $f_i : M_{i-1} \to M_i$ as a bunch of short exact sequences $0 \to N_i \to M_i \to N_{i+1} \to 0$, where $N_j = im(f_j) = ker(f_{j+1})$, $N_i \to M_i$ is the inclusion map, $M_i \to N_{i+1}$ is just f_{i+1} with its codomain restricted.

We would like to view the S^{-1} construction as a functor from the category of R-modules to the category of $S^{-1}R$-modules. We already defined $S^{-1}M$ and made it into an $S^{-1}R$-module. Now if $\gamma : M \to N$ is an R-linear map we attempt to define $S^{-1}\gamma : S^{-1}M \to S^{-1}N$ by $S^{-1}\gamma : m/s \mapsto \gamma(m)/s$.

Routinely: $S^{-1}\gamma$ is well-defined, $S^{-1}(id_M) = id_{S^{-1}M}$, $S^{-1}(\gamma \circ \delta) = S^{-1}\gamma \circ S^{-1}\delta$. This is enough to see it is a functor. Actually it has other nice properties: we will use the (easy!) facts that $S^{-1}0 = 0$ and if γ is the zero map from M to N then $S^{-1}\gamma$ is the zero map from $S^{-1}M$ to $S^{-1}N$.

Theorem 1. *If $f_i : M_{i-1} \to M_i$ is an exact sequence of R-modules then $S^{-1}f_i : S^{-1}M_{i-1} \to S^{-1}M_i$ is an exact sequence of $S^{-1}R$-modules.*

Proof. Since $f_{i+1} \circ f_i = 0$, it follows from the properties of the S^{-1} functor that $S^{-1}f_{i+1} \circ S^{-1}f_i = 0$, where we abuse notation and use "0" as a generic name for a map which is constantly zero. So the image of $S^{-1}f_i$ is contained in the kernel of $S^{-1}f_{i+1}$.

Conversely let $m/s \in ker(S^{-1}f_{i+1})$, so by definition $f_{i+1}(m)/s = 0$ in $S^{-1}M_{i+1}$. By definition there is $u \in S$ with $uf_{i+1}(m) = 0$, and by the R-linearity of f_{i+1} we have $f_{i+1}(um) = 0$, so that by exactness of our original sequence $um \in ker(f_{i+1}) = im(f_i)$. Fix $b \in M_{i-1}$ with $f_i(b) = um$, and define $v \in S^{-1}M_{i-1}$ by $v = b/us$. Then by definition $S^{-1}f_i(v) = f_i(b)/us = um/us = m/s$, so that $m/s \in im(S^{-1}f_i)$ as required. □

As a sample application we study the interaction of S^{-1} with the quotient module construction. Suppose that M and N are R-modules with $M \leq N$ and consider the short exact sequence $0 \to M \to N \to N/M \to 0$ with the usual inclusion and quotient maps. By the work above $S^{-1}0 = 0 \to S^{-1}M \to S^{-1}N \to S^{-1}N/M \to S^{-1}0 = 0$ is short exact. So we may conclude that $S^{-1}M \leq S^{-1}N$ (contrast the nasty situation with tensor products!) and that $S^{-1}N/M \simeq S^{-1}N/S^{-1}M$ as $S^{-1}R$-modules.