# CA LECTURE 11

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By the discussion from the end of last time: in general when $M \leq M'$ and $N \leq N'$ are $R$-modules there is a natural HM from $M \otimes N$ to $M' \otimes N'$ which maps $m \otimes n$ (as an element of $M \otimes N$) to $m \otimes n$ (as an element of $M' \otimes N'$). This may not be injective. However we do know that if $m \otimes n = 0$ in $M \otimes N$ then $m \otimes n = 0$ in $M' \otimes N'$. The next result is a partial converse.

**Fact 1.** *If $m \otimes n = 0$ in $M \otimes N$ there exists fg submodules $M_1 \leq M$ and $N_1 \leq N$ such that $m \in M_1$, $n \in N_1$ and $m \otimes n = 0$ in $M_1 \otimes N_1$.*

*Proof.* By definition of $M \otimes N$, $m \otimes n = (m, n) + X$ for a certain submodule $X$ of $Fr_R(M \times N)$. So since $m \otimes n = 0$ we have $(m, n) \in X$, and can write $(m, n)$ as a finite $R$-linear combination of the generators of $X$, say $\sum_{i=1}^{k} r_i x_i$. Now let $M_1$ be the submodule generated by $m$ and all elements of $M$ which are mentioned among the $x_i$, and define $N_1$ similarly. Then the equation $(m, n) = \sum_{i=1}^{k} r_i x_i$ holds in the module $Fr_R(M_1 \times N_1)$ and witnesses that $m \otimes n = 0$ in $M_1 \otimes N_1$. □

Note: Saying that $m \otimes n = 0$ in $M \otimes N$ is equivalent to saying that $\phi(m, n) = 0$ for all bilinear maps $\phi$ on $M \times N$.

Now we study a variation on the theory of modules.

**Definition 1.** *Let $R$ and $S$ be rings. An $(R, S)$-bimodule is a set $M$ equipped with a binary operation $+$ and two "scalar multiplications" $R \times M \to M$ and $S \times M \to M$ such that*

1. *$M$ is an $R$-module.*
2. *$M$ is an $S$-module.*
3. *$r(sm) = s(rm)$ for all $r \in R$, $s \in S$ and $m \in M$.*

Example: recall that an $R$-*algebra* is a ring $S$ equipped with a HM $\phi : R \to S$. It is easy to see that $S$ is an $(R, S)$-bimodule where the $R$-scalar multiplication is $(r, s) \mapsto \phi(r) \times_S s$.

Now we show that if $M$ is an $R$-module and $N$ is an $(R, S)$-bimodule then $M \otimes_R N$ can be given an $(R, S)$-bimodule structure. We write the subscript on $\otimes_R$ to stress that we are computing the $\otimes$ in the category of $R$-modules using the $R$-module structure on $S$.

For a fixed $s \in S$ consider the map $(m, n) \mapsto m \otimes (sn)$ from $M \times N$ to $M \otimes_R N$. It is easy to see it is $R$-bilinear, for example

$$m \otimes s(rn) = m \otimes r(sn) = r(m \otimes sn).$$

So by the usual argument there is a unique $R$-linear map from $M \otimes N$ to $M \otimes N$ taking $m \otimes n$ to $m \otimes sn$; this will define for us the operation of scalar multiplication by the element $s$. It is routine to check that the $(R, S)$-bimodule axioms are satisfied.

Most important special case: Let $M$ be an $R$-module and $S$ an $R$-algebra, then we can give $M \otimes_R S$ the structure of an $S$-module. This is sometimes called "change of base".

We do an exercise (2.15) from A and M to firm up these ideas. Let $R$ and $S$ be rings and let $M$ be an $R$-module, $N$ an $(R, S)$-bimodule and $P$ an $S$-module. Then we claim that $(M \otimes_R N) \otimes_S P \simeq M \otimes_R (N \otimes_S P)$ as $(R, S)$-bimodules, via an IM in which $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$.

We consider the category of maps $\phi$ such that the domain of $\phi$ is $M \times N \times P$, the codomain is some $(R, S)$-bimodule and $\phi$ is $R$-linear in its first and second arguments and $S$-linear in its second and third arguments. A morphism from $\phi_1 : M \times N \times P \to Q_1$ to $\phi_1 : M \times N \times P \to Q_2$ will (of course) be a map $\gamma : Q_1 \to Q_2$ which is both $R$-linear and $S$-linear (these are the morphisms in the category of bimodules).

We claim that $(a, b, c) \mapsto a \otimes (b \otimes c)$ and $(a, b, c) \mapsto (a \otimes b) \otimes c$ are both initial elements in this category. It is routine to check the multilinearity of these maps so they are objects of our category. Suppose that $\phi : M \times N \times P \to Q$ is an arbitrary object. For $p$ fixed the map $(m, n) \mapsto \phi(m, n, p)$ is $R$-bilinear so we there is a unique $\phi_1 : (M \otimes_R N) \times P \to Q$ such that $\phi_1 : (m \otimes n, p) \mapsto \phi(m, n, p)$, and $\phi_1$ is $R$-linear in its first argument. It is routine to see $\phi_1$ is $S$-bilinear for example

$$\phi_1(s(m \otimes n), p) = \phi_1(m \otimes sn, p) = \phi(m, sn, p) = s\phi(m, np) = s\phi_1(m \otimes n, p).$$

So there is a unique $\phi_2 : (M \otimes_R N) \otimes_S P$ such that $\phi_2$ is $S$-linear and $\phi_2 : (m \otimes n) \otimes p \mapsto \phi(m, n, p)$. It is routine to check that $\phi_2$ is also $R$-linear. A similar argument works for $(m, n, p) \mapsto m \otimes (n \otimes p)$ and we are done by uniqueness of initial objects.

Suppose we have a ring $R$, an MC set $S \subseteq R$ and an $R$-module $M$. We now have two ways to manufacture an $S^{-1}R$-module, namely we can either form $S^{-1}M$ or $S^{-1}R \otimes_R M$. We see next time that these two constructions are equivalent.