CA LECTURE 11

SCRIBE: CLAIRE TOMESCH

By the discussion from the end of last time: in general when $M \leq M'$ and $N \leq N'$ are R-modules there is a natural HM from $M \otimes N$ to $M' \otimes N'$ which maps $m \otimes n$ (as an element of $M \otimes N$) to $m \otimes n$ (as an element of $M' \otimes N'$). This may not be injective. However we do know that if $m \otimes n = 0$ in $M \otimes N$ then $m \otimes n = 0$ in $M' \otimes N'$. The next result is a partial converse.

Fact 1. *If $m \otimes n = 0$ in $M \otimes N$ there exists fg submodules $M_1 \leq M$ and $N_1 \leq N$ such that $m \in M_1$, $n \in N_1$ and $m \otimes n = 0$ in $M_1 \otimes N_1$.*

Proof. By definition of $M \otimes N$, $m \otimes n = (m, n) + X$ for a certain submodule X of $Fr_R(M \times N)$. So since $m \otimes n = 0$ we have $(m, n) \in X$, and can write (m, n) as a finite R-linear combination of the generators of X, say $\sum_{i=1}^{k} r_i x_i$. Now let M_1 be the submodule generated by m and all elements of M which are mentioned among the x_i, and define N_1 similarly. Then the equation $(m, n) = \sum_{i=1}^{k} r_i x_i$ holds in the module $Fr_R(M_1 \times N_1)$ and witnesses that $m \otimes n = 0$ in $M_1 \otimes N_1$. □

Note: Saying that $m \otimes n = 0$ in $M \otimes N$ is equivalent to saying that $\phi(m, n) = 0$ for all bilinear maps ϕ on $M \times N$.

Now we study a variation on the theory of modules.

Definition 1. *Let R and S be rings. An (R, S)-bimodule is a set M equipped with a binary operation $+$ and two "scalar multiplications" $R \times M \to M$ and $S \times M \to M$ such that*

1. *M is an R-module.*
2. *M is an S-module.*
3. *$r(sm) = s(rm)$ for all $r \in R$, $s \in S$ and $m \in M$.*

Example: recall that an *R-algebra* is a ring S equipped with a HM $\phi : R \to S$. It is easy to see that S is an (R, S)-bimodule where the R-scalar multiplication is $(r, s) \mapsto \phi(r) \times_S s$.

Now we show that if M is an R-module and N is an (R, S)-bimodule then $M \otimes_R N$ can be given an (R, S)-bimodule structure. We write the subscript on $\otimes_R$ to stress that we are computing the $\otimes$ in the category of R-modules using the R-module structure on S.

For a fixed $s \in S$ consider the map $(m, n) \mapsto m \otimes (sn)$ from $M \times N$ to $M \otimes_R N$. It is easy to see it is R-bilinear, for example

$$m \otimes s(rn) = m \otimes r(sn) = r(m \otimes sn).$$

So by the usual argument there is a unique R-linear map from $M \otimes N$ to $M \otimes N$ taking $m \otimes n$ to $m \otimes sn$; this will define for us the operation of scalar multiplication by the element s. It is routine to check that the (R, S)-bimodule axioms are satisfied.

Most important special case: Let M be an R-module and S an R-algebra, then we can give $M \otimes_R S$ the structure of an S-module. This is sometimes called "change of base".

We do an exercise (2.15) from A and M to firm up these ideas. Let R and S be rings and let M be an R-module, N an (R, S)-bimodule and P an S-module. Then we claim that $(M \otimes_R N) \otimes_S P \simeq M \otimes_R (N \otimes_S P)$ as (R, S)-bimodules, via an IM in which $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$.

We consider the category of maps ϕ such that the domain of ϕ is $M \times N \times P$, the codomain is some (R, S)-bimodule and ϕ is R-linear in its first and second arguments and S-linear in its second and third arguments. A morphism from $\phi_1 : M \times N \times P \to Q_1$ to $\phi_1 : M \times N \times P \to Q_2$ will (of course) be a map $\gamma : Q_1 \to Q_2$ which is both R-linear and S-linear (these are the morphisms in the category of bimodules).

We claim that $(a, b, c) \mapsto a \otimes (b \otimes c)$ and $(a, b, c) \mapsto (a \otimes b) \otimes c$ are both initial elements in this category. It is routine to check the multilinearity of these maps so they are objects of our category. Suppose that $\phi : M \times N \times P \to Q$ is an arbitrary object. For p fixed the map $(m, n) \mapsto \phi(m, n, p)$ is R-bilinear so we there is a unique $\phi_1 : (M \otimes_R N) \times P \to Q$ such that $\phi_1 : (m \otimes n, p) \mapsto \phi(m, n, p)$, and ϕ_1 is R-linear in its first argument. It is routine to see ϕ_1 is S-bilinear for example

$$\phi_1(s(m \otimes n), p) = \phi_1(m \otimes sn, p) = \phi(m, sn, p) = s\phi(m, np) = s\phi_1(m \otimes n, p).$$

So there is a unique $\phi_2 : (M \otimes_R N) \otimes_S P$ such that ϕ_2 is S-linear and $\phi_2 : (m \otimes n) \otimes p \mapsto \phi(m, n, p)$. It is routine to check that ϕ_2 is also R-linear. A similar argument works for $(m, n, p) \mapsto m \otimes (n \otimes p)$ and we are done by uniqueness of initial objects.

Suppose we have a ring R, an MC set $S \subseteq R$ and an R-module M. We now have two ways to manufacture an $S^{-1}R$-module, namely we can either form $S^{-1}M$ or $S^{-1}R \otimes_R M$. We see next time that these two constructions are equivalent.