CA LECTURE 10

SCRIBE: SPAS BOJANOV

Bilinearity: let R be a ring and A, B, C be R-modules. A function ϕ from $A \times B \to C$ is *bilinear* iff $a \mapsto \phi(a,b)$ is a linear map for each b and $b \mapsto \phi(a,b)$ is a linear map for each a. Here "linear map" is synonymous with "R-module HM".

It is easy to see that if ϕ is bilinear then $\phi(0,b) = \phi(a,0) = 0$. Also if $\phi : A \times B \to C$ is bilinear and $\alpha : C \to D$ is linear then $\alpha \circ \phi : A \times B \to D$ is bilinear.

Given R, A, B we consider a category whose objects are bilinear maps on $A \times B$. If $\phi_1 : A \times B \to C_1$ and $\phi_2 : A \times B \to C_2$ are bilinear, a morphism from ϕ_1 to ϕ_2 is just $\alpha : C_1 \to C_2$ such that $\alpha \circ \phi_1 = \phi_2$. We will construct an initial object in this category, the *tensor product* of A and B.

Recall: if R is a ring and X is a set then the *free R-module* $Fr_R(x)$ is the set of all functions $f : X \to R$ such that $\{x : f(x) \neq 0\}$ is finite, made into a module by defining addition and scalar multiplication coordinate-wise. We identify $x \in X$ with the function in $Fr(X)$ which takes value 1 at x and 0 elsewhere. Then we may write a typical element of $Fr(X)$ as $\sum_i \lambda_i x_i$ for some $\lambda_i \in R$ and $x_i \in X$.

The key property: if R is a ring, X is a set and M is an R-module then for any function $f : X \to M$ there is a unique R-module HM $\phi : Fr(X) \to M$ such that $\phi \restriction X = f$. ϕ is given (as it must be) by $\phi : \sum_i \lambda_i x_i \mapsto \sum_i \lambda_i f(x_i)$.

Given R, A, B let $X = A \times B$ and $M = Fr(X)$. $X \subseteq M$ but the inclusion map $X \hookrightarrow M$ is typically not bilinear. We will quotient out by the least submodule N with the property that the composition $X \hookrightarrow M \to M/N$ of the inclusion and quotient maps is bilinear.

Let $N \leq M$ be the submodule generated by all elements in M of the form

$$(a_1+a_2, b)-(a_1, b)-(a_2, b), (a, b_1+b_2)-(a, b_1)-(a, b_2), (ra, b)-r(a, b), (a, rb)-r(a, b).$$

We define $A \otimes B = M/N$ and a function $\otimes : A \times B \to A \otimes B$ by $\otimes : (a,b) \mapsto (a,b)+N$. We write $a \otimes b$ for $\otimes(a,b)$. We will show that $\otimes$ is bilinear and is initial in the category of bilinear maps on $A \times B$.

Claim 1: $\otimes$ is bilinear. The proof is easy!

Claim 2: $\otimes$ is an initial object. Explicitly for every C and every bilinear $\phi : A \times B \to C$ there is a unique linear $\alpha : A \otimes B \to C$ with $\alpha \circ \otimes = \phi$, that is to say $\alpha(a \otimes b) = \phi(a,b)$ for all a and b.

Existence: extend ϕ to an R-module HM $\bar{\phi}$ from M to C by $\bar{\phi} : \sum_i \lambda_i (a_i, b_i) \mapsto \sum_i \lambda_i \phi(a_i, b_i)$. Note that

$$\bar{\phi}((a_1 + a_2, b) - (a_1, b) - (a_2, b)) = \phi(a_1 + a_2, b) - \phi(a_1, b) - \phi(a_2, b) = 0,$$

as ϕ is bilinear. Similarly for the other generators of N so $N \leq \ker(\phi) \leq M$. So we may define a linear map α from $M/N = A \otimes B$ to C by $\alpha : m + N \mapsto \bar{\phi}(m)$. What is more

$$\alpha(a \otimes b) = \alpha((a,b) + M) = \bar{\phi}((a,b)) = \phi(a,b),$$

so that α is as required.

Digression on the nature of $A \otimes B$: We claim elements of $A \otimes B$ have the form $\sum_{i=1}^n a_i \otimes b_i$. To see this let $m = \sum_{i=1}^n \lambda_i(c_i, d_i)$. Then

$$m + N = \sum_{i=1}^n \lambda_i(c_i \otimes d_i) = \sum_{i=1}^n (\lambda_i c_i) \otimes d_i,$$

or equivalently $\sum_{i=1}^n c_i \otimes (\lambda_i d_i)$. IMPORTANT: this representation is typically not unique.

Uniqueness of α: if α exists it can be computed by the formula $\alpha : \sum_i a_i \otimes b_i \mapsto \sum_i \phi(a_i, b_i)$.

To illustrate the ideas we show that $R \otimes M \simeq M$. Idea: we show that the map $\psi : (r, m) \mapsto rm$ from $R \times M$ to M is initial in the categpry of bilinear maps on $R \times M$.

So suppose that $\phi : R \times M \mapsto C$ is bilinear. Then $\phi(r, m) = \phi(r1, m) = r\phi(1, m) = \phi(1, rm)$. If we define $\alpha : M \to C$ by $\alpha : m \mapsto \phi(1, m)$ then

(1) α is linear, since ϕ is bilinear.
(2) $\alpha \circ \psi = \phi$ because $\alpha(\psi(r, m)) = \alpha(rm) = \phi(1, rm) = \phi(r, m)$.
(3) If $\beta \circ \psi = \phi$ then in particular $\beta(m) = \beta(\psi(1, m)) = \phi(1, m)$, so $\beta = \alpha$.

By the uniqueness of initial objects there is an IM $\beta : R \otimes M \simeq M$ such that $\beta : r \otimes m \mapsto rm$, equivalently $\beta : \sum_i r_i \otimes m_i \mapsto \sum_i r_i m_i$.

Some other important IMs: $M \otimes N \simeq N \otimes M$ via a map $m \otimes n \mapsto n \otimes m$, and $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ via a map $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.

Suppose that $\phi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ are HMs of R-modules. Then the map $(m_1, n_1) \mapsto \phi(m_1) \otimes \psi(n_1)$ is bilinear, so there is a unique linear map $\alpha : M_1 \otimes N_1 \to M_2 \otimes N_2$ such that $\alpha : m_1 \otimes n_1 \mapsto \phi(m_1) \otimes \psi(n_1)$ This map is usually called $\phi \otimes \psi$.

Cautionary example: let $R = \mathbb{Z}$ so R-modules are just abelian groups. Then $\mathbb{Z} \leq \mathbb{Q}$ but if $G = \mathbb{Z}/2\mathbb{Z}$ then $\mathbb{Z} \otimes G \simeq G$ while $\mathbb{Q} \otimes G = \{0\}$. The latter equation is obtained by noting that $q \otimes 1 = (2q/2) \otimes 1 = q/2 \otimes 0 = 0$.