COMMUTATIVE ALGEBRA HW 9 SOLUTIONS

JC

Due in class Mon 26 September.

(1) Prove that for every ring R there is a unique ring HM from $\mathbb{Z}$ to R.

Routine.

(2) By Q1 every ring has a unique $\mathbb{Z}$-algebra structure (in rather the same way that every abelian group has a unique $\mathbb{Z}$-module structure, so that we blur the distinction).

Let R and S be rings. Show that $R \otimes_{\mathbb{Z}} S$ can be made into a ring in such a way that $(r_1 \otimes s_1) \times (r_2 \otimes s_2) = r_1 r_2 \otimes s_1 s_2$. You should verify the ring axioms! (Look on p30 of A and M if you get stuck).

Consider the map from $R \times S \times R \times S$ to $R \otimes_{\mathbb{Z}} S$ given by $(r_1, s_1, r_2, s_2) \mapsto r_1 r_2 \otimes s_1 s_2$. It is easily seen to be $\mathbb{Z}$-linear in each argument, so holding r_1 and s_1 constant we get a bilinear map: hence by the usual argument there is a function from $R \times S \times (R \otimes S)$ to $R \otimes S$ such that $(r_1, s_1, r_2 \otimes s_2) \mapsto r_1 r_2 \otimes s_1 s_2$ and the function is linear in its third argument. This is easily seen to be linear in its other arguments as well so we get a map from $(R \otimes S) \times (R \otimes S)$ to $R \otimes S$ which is bilinear and $(r_1 \otimes s_1, r_2 \otimes s_2) \mapsto r_1 r_2 \otimes s_1 s_2$

It is now routine (if unpleasnat) to check the ring axioms. Points to keep track of: a general element of $R \otimes S$ is a finite sum $\sum_{i=1}^{n} r_i \otimes s_i$, and the $\mathbb{Z}$-bilinearity of the multiplication is exactly what is needed for the distributive law.

Cultural note: A and M do something equivalent. Explicitly they start with the same map $(r_1, s_1, r_2, s_2) \mapsto r_1 r_2 \otimes s_1 s_2$, make it into a linear map from a "multi tensor product" (see discussion at top of p26) $R \otimes S \otimes R \otimes S$ to $R \otimes S$, and then use an IM between $R \otimes S \otimes R \otimes S$ and $(R \otimes S) \otimes (R \otimes S)$ to get a bilinear map from $(R \otimes S) \times (R \otimes S)$.

(3) Verify that the ring $R \otimes S$ together with the maps $r \mapsto r \otimes 1$ and $s \mapsto 1 \otimes s$ constitute a coproduct in the category of rings. Hint: given $\phi : R \to T$ and $\psi : S \to T$ how can we cook up a $\mathbb{Z}$-bilinear map from $R \times S$ to T?

It is routine to check that $r \mapsto r \otimes 1$ and $s \mapsto 1 \otimes s$ are HMs. Suppose that $\phi : R \to T$ and $\psi : S \to T$ are ring HMs and define a $\mathbb{Z}$-bilinear map from $R \times S$ to T by $(r, s) \mapsto \phi(r)\psi(s)$. As usual we have a unique $\mathbb{Z}$-linear map $\gamma : R \otimes_{\mathbb{Z}} S \to T$ such that $\gamma : r \otimes s \mapsto \phi(r)\psi(s)$, and it is routine to check that γ actually has the stronger property of being a ring HM. Also $\gamma(r \otimes 1) = \phi(r)$ and $\gamma(1 \otimes s) = \psi(s)$ so that γ makes the required diagram commute. Finally if $\delta : R \otimes S \to T$ is any ring HM such that $\delta(r \otimes 1) = \phi(r)$ and $\delta(1 \otimes s) = \psi(s)$ then δ is $\mathbb{Z}$-linear and also since $r \otimes s = (r \otimes 1)(1 \otimes s)$ we get $\delta(r \otimes s) = \phi(r)\psi(s)$, so that $\delta = \gamma$ by the uniqueness properties of γ.