COMMUTATIVE ALGEBRA HW 7 SOLUTIONS

JC

Due in class Wed 21 September.

(1) (A and M 1.19) A topological space is said to be *irreducible* iff it is nonempty and every pair of nonempty open sets has a nonempty intersection. Let R be a ring. Show that $Spec(R)$ is irreducible iff the nilradical is prime.

Note that $Spec(R) \neq \emptyset$ iff R is not the zero ring. We assume without loss of generality that this is the case.

Since open sets are unions of the sets O_a, the space is irreducible iff every two nonempty O_a's intersect. Now $O_a = \emptyset$ iff $P \notin O_a$ for every prime P iff $a \in P$ for every prime P iff a is nilpotent. Since $O_a \cap O_b = O_{ab}$, the space is irreducible iff for all non-nilpotent a and b the product ab is non-nilpotent, which is exactly the condition that the nilradical be prime in a nonzero ring.

(2) (A and M 3.1 with a twist) Let M be an fg R-module and let $S \subseteq R$ be a MC set. Show that $S^{-1}M = \{0\}$ iff there is $s \in S$ such that $sM = \{0\}$. What if M is not fg?

Start by noting that $m/s = 0$ iff $m/s = 0/1$ iff there is $u \in S$ such that $um = 0$, so that $S^{-1}M = 0$ iff every element of M is annihilated by some member of S.

Clearly if $s \in S$ annihilates all of M then $S^{-1}M = \{0\}$. Conversely if M is fg with generating set $m_1, \dots m_n$ and $S^{-1}M = 0$ then we may choose $s_i \in S$ such that $s_i m_i = 0$, and if $s = s_1 \dots s_n$ then easily $sm = 0$ for all M.

Suppose now that $R = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$ (as an abelian group, that is a $\mathbb{Z}$-module) and $S = \mathbb{Z} \setminus \{0\}$. Then every element of M is annihilated by some element of S but no element of S annihilates all of M.

(3) Let R be a ring. Show that every minimal prime ideal of R is contained in the set of zero-divisors of R.

Let P be minimal prime so that by an old HW $T = R \setminus P$ is maximal among MC sets S with $0 \notin S$. Let $a \in P$, then the least MC set constining T and a is $T \cup aT \cup a^2T \dots$, so that we have $a^n b = 0$ for some $n > 0$ and $b \in T$. If n is least with $a^n b = 0$ then $a^{n-1} b \neq 0$ so a is a zero-divisor.