COMMUTATIVE ALGEBRA HW 6 SOLUTIONS

JC

Due in class Mon 19 September.

(1) A map $f : X \to Y$ between topological spaces is called *continuous* iff for every open subset O of Y, $f^{-1}[O]$ is open in X. The category **Top** has the topological spaces as objects and the continuous maps as morphisms.

$Spec(R)$ is the set of prime ideals of R with the topology described in a previous HW, and for each ring HM $f : R \to S$ we define $Spec(f) : Spec(S) \to Spec(R)$ by $Spec(f) : P \mapsto f^{-1}[P]$ for each $P \in Spec(S)$. Show that $Spec(f)$ is continuous, $Spec(id_R) = id_{Spec(R)}$ and $Spec(g \circ f) = Spec(f) \circ Spec(g)$.

(This is an example of a "contravariant functor" from rings to spaces).

Continuity: it is enough to show that the inverse image of each set O_r is open. Now

$$Q \in Spec(f)^{-1}[O_r] \iff f^{-1}[Q] \in O_r \iff r \notin f^{-1}[Q] \iff f(r) \notin Q \iff Q \in O_{f(r)},$$

so we are done. The other claims are easy.

(2) Let A be a ring, let I be an ideal of A and S a multiplicatively closed subset of A. Let $\bar{S}$ be the image of S in the quotient ring A/I under the usual quotient map and let $S^{-1}I$ be the extension of I in $S^{-1}A$, so that $S^{-1}I$ is the set of a/s with $a \in I$, $s \in S$.

Prove that the ring of fractions $\bar{S}^{-1}A/I$ is isomorphic to the quotient ring $S^{-1}A/S^{-1}I$.

Many solutions are possible: we give a rather conceptual one. Recall that

(a) The HM $a \mapsto a + I$ from A to A/I is such that for every ring HM $\phi : A \to B$ such that $\phi(a) = 0$ for all $a \in I$, there is a unique HM $\alpha : A/I \mapsto B$ such that $\alpha(a + I) = \phi(a)$. To put it another way $a \mapsto a + I$ is initial in the category of HMs with domain A which are zero on I.

(b) The HM $a \mapsto a/1$ from A to $S^{-1}A$ is such that for every ring HM $\phi : A \to B$ such that $\phi(s)$ is a unit for all $a \in S$, there is a unique $\alpha : S^{-1}A \mapsto B$ such that $\alpha(a/1) = \phi(a)$.

To put it another way $a \mapsto a/1$ is initial in the category of HMs with domain A which take unit values on S.

Now consider a ring HM $\phi : A \to B$ which both takes the value zero on I and unit values on S. Applying 1 above there is a unique $\psi : A/I \to B$ such that $\psi : a+I \mapsto \phi(a)$. Since ψ takes unit values on $\bar{S}$ by 2 above there is a unique $\rho : \bar{S}^{-1}A/I \to B$ such that $\rho : (a+I)/(1+I) \mapsto \phi(a)$. To put it another way the map $a \mapsto (a+I)/(1+I)$ is initial in the category of maps which are zero on I and take unit values on S.

Now take such a ϕ and apply 2 then 1. By 2 there is $\psi^* : S^{-1}A \to B$ such that $\psi^* : a/1 \mapsto \phi(a)$, and then by 1 since ψ^* vanishes on $S^{-1}I = I^e$ there is $\rho^* : S^{-1}A/S^{-1}I \to B$ such that $\rho^* : a/1 + S^{-1}A \mapsto \phi(a)$. So the map $a \mapsto a/1 + S^{-1}I$ is also initial in the category of maps which are zero on I and take unit values on S.

By the uniqueness of initial objects there is a unique IM between $\bar{S}^{-1}A/I$ and $S^{-1}A/S^{-1}I$ mapping $(a+I)/(1+I)$ to $a/1 + S^{-1}I$. So clearly an explicit formula for this IM is $(a+I)/(s+I) \mapsto a/s + S^{-1}I$.

I like this proof because it explains the IM between the two rings but other proofs are possible: for example consider the map $a/s \mapsto (a+I)/(s+I)$, show it's a well-defined surjective HM with kernel $S^{-1}I$. Or just guess the IM given above and check it works.

(3) A subset S of a ring R is *saturated* iff $1 \in S$ and $ab \in S$ iff $a \in S$ and $b \in S$ for all a, b. Prove that a set S is saturated iff $R \setminus S$ is a (possibly empty) union of prime ideals.

It is easy to see that the complement of a prime ideal is saturated and that an intersection of saturated sets is saturated: so the complement of a union of primes is saturated.

For the other direction let S be saturated and let $a \notin S$. Note that then $0 \notin S$ because $0 = r0 \in S \implies r \in S$ for all $r \in R$ so that R is the only saturated set containing 0. Now clearly (a) is disjoint from S because $ra \in S \implies a \in S$, so by ZL we may extend (a) to an ideal which is maximal in the set of ideals disjoint from S, and such an ideal is prime. So every $a \notin S$ is in a prime ideal disjoint from S, hence the complement of S is a union of primes.

(4) Prove that the set of zero-divisors in a ring is a union of prime ideals. Give an example of a ring with some non-nilpotent zero-divisors and show how to write the set of zero-divisors in your example as a union of prime ideals.

We claim the set of non-ZDs is saturated. If ab is not a ZD then $abr \neq 0$ for all $r \neq 0$, so that $ar, br \neq 0$ and hence a, b are not ZDs. Conversely if a and b are not ZDS then $r \neq 0 \implies br \neq 0 \implies abr \neq 0$, so that ab is not a ZD.

Example: in $\mathbb{Z}/6\mathbb{Z}$ we have $\{0, 2, 3\} = \{0, 2\} \cup \{0, 3\}$.