COMMUTATIVE ALGEBRA HW 6

JC

Due in class Mon 19 September.

(1) A map $f : X \to Y$ between topological spaces is called *continuous* iff for every open subset O of Y, $f^{-1}[O]$ is open in X. The category **Top** has the topological spaces as objects and the continuous maps as morphisms.

$Spec(R)$ is the set of prime ideals of R with the topology described in a previous HW, and for each ring HM $f : R \to S$ we define $Spec(f) : Spec(S) \to Spec(R)$ by $Spec(f) : P \mapsto f^{-1}[P]$ for each $P \in Spec(S)$. Show that $Spec(f)$ is continuous, $Spec(id_R) = id_{Spec(R)}$ and $Spec(g \circ f) = Spec(f) \circ Spec(g)$.

(This is an example of a "contravariant functor" from rings to spaces).

(2) Let A be a ring, let I be an ideal of A and S a multiplicatively closed subset of A. Let $\bar{S}$ be the image of S in the quotient ring A/I under the usual quotient map and let $S^{-1}I$ be the extension of I in $S^{-1}A$, so that $S^{-1}I$ is the set of a/s with $a \in I$, $s \in S$.

Prove that the ring of fractions $\bar{S}^{-1}A/I$ is isomorphic to the quotient ring $S^{-1}A/S^{-1}I$.

(3) A subset S of a ring R is *saturated* iff $1 \in S$ and $ab \in S$ iff $a \in S$ and $b \in S$ for all a, b. Prove that a set S is saturated iff $R \setminus S$ is a (possibly empty) union of prime ideals.

(4) Prove that the set of zero-divisors in a ring is a union of prime ideals. Give an example of a ring with some non-nilpotent zero-divisors and show how to write the set of zero-divisors in your example as a union of prime ideals.