

## COMMUTATIVE ALGEBRA HW 6

JC

Due in class Mon 19 September.

- (1) A map  $f : X \rightarrow Y$  between topological spaces is called *continuous* iff for every open subset  $O$  of  $Y$ ,  $f^{-1}[O]$  is open in  $X$ . The category **Top** has the topological spaces as objects and the continuous maps as morphisms.

$\text{Spec}(R)$  is the set of prime ideals of  $R$  with the topology described in a previous HW, and for each ring homomorphism  $f : R \rightarrow S$  we define  $\text{Spec}(f) : \text{Spec}(S) \rightarrow \text{Spec}(R)$  by  $\text{Spec}(f) : P \mapsto f^{-1}[P]$  for each  $P \in \text{Spec}(S)$ . Show that  $\text{Spec}(f)$  is continuous,  $\text{Spec}(id_R) = id_{\text{Spec}(R)}$  and  $\text{Spec}(g \circ f) = \text{Spec}(f) \circ \text{Spec}(g)$ .

(This is an example of a “contravariant functor” from rings to spaces).

- (2) Let  $A$  be a ring, let  $I$  be an ideal of  $A$  and  $S$  a multiplicatively closed subset of  $A$ . Let  $\bar{S}$  be the image of  $S$  in the quotient ring  $A/I$  under the usual quotient map and let  $S^{-1}I$  be the extension of  $I$  in  $S^{-1}A$ , so that  $S^{-1}I$  is the set of  $a/s$  with  $a \in I$ ,  $s \in S$ .

Prove that the ring of fractions  $\bar{S}^{-1}A/I$  is isomorphic to the quotient ring  $S^{-1}A/S^{-1}I$ .

- (3) A subset  $S$  of a ring  $R$  is *saturated* iff  $1 \in S$  and  $ab \in S$  iff  $a \in S$  and  $b \in S$  for all  $a, b$ . Prove that a set  $S$  is saturated iff  $R \setminus S$  is a (possibly empty) union of prime ideals.
- (4) Prove that the set of zero-divisors in a ring is a union of prime ideals. Give an example of a ring with some non-nilpotent zero-divisors and show how to write the set of zero-divisors in your example as a union of prime ideals.