COMMUTATIVE ALGEBRA HW 4

JC

Due in class Mon 12 September.

(1) (A and M III.6) Let $A \neq \{0\}$ be a ring. let Σ be the set of all multiplicatively closed sets S with $0 \notin S$. Show that
  (a) Σ has maximal elements.
  (b) S is maximal in Σ iff $A \setminus S$ is a minimal prime ideal.

(2) (A and M I.12) Recall that a *local ring* is a ring with a unique maximal ideal, equivalently a ring where the non-units form an ideal. Recall also that e is *idempotent* iff $e^2 = e$. Show that in a local ring the only idempotents are 0 and 1.

(3) True or false? If $R[x]$ is Noetherian then R is Noetherian.

(4) A *topological space* is a set X equipped with a collection $\mathcal{O}$ of subsets of X (the "open sets") satisfying the axioms:
  (a) $\emptyset$ and X are in $\mathcal{O}$.
  (b) The intersection of any two elements of $\mathcal{O}$ is an element of $\mathcal{O}$.
  (c) The union of any set of elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

  (A homely example: Let (X, d) be a metric space and $\mathcal{O}$ be the set of open sets for this metric)

  Let R be an arbitrary ring and let $Spec(R)$ (the *spectrum of* R) be the set of prime ideals of R. For each $a \in R$ let $O_a = \{P \in Spec(R) : a \notin P\}$. Define $\mathcal{O}$ to be the set of all subsets of $Spec(R)$ which are unions of sets of the form O_a, or more explicitly $X \in \mathcal{O}$ iff for every $P \in X$ there is $a \in R$ so that $P \in O_a \subseteq X$.

  Show that this choice of $\mathcal{O}$ makes $Spec(R)$ into a topological space.