COMMUTATIVE ALGEBRA HW 3 SOLUTIONS

JC

(1) Read the "Categories and products" handout on the course webpage.

(2) Let R be a ring and define a category $\mathcal{C}$ as follows. The objects are all triples (S, a, ϕ) where S is a ring, $\phi : R \to S$ is a ring HM and $a \in S$. The morphisms from (S_1, a_1, ϕ_1) to (S_2, a_2, ϕ_2) are all ring HMs $\psi : S_1 \to S_2$ such that $\psi \circ \phi_1 = \phi_2$ and $\psi(a_1) = a_2$.

Show that $(R[x], x, \phi)$ is an initial object in this category, where ϕ is the usual inclusion map from R to $R[x]$.

Suppose that (S, a, ψ) is any object in the category, so by definition $\psi : R \to S$ and $a \in S$. If there is a HM α from $R[x]$ to S such that $\alpha(x) = a$ and $\alpha \circ \phi = \psi$, then clearly α must be given by

$$\alpha(\sum_i r_i x^i) = \sum_i \psi(r_i) a^i.$$

It is routine to check that this formula does define a ring HM.

(3) Let $\mathcal{C}$ be a category and let a and b be objects in $\mathcal{C}$. A *coproduct* of a and b is an object c together with maps $f : a \to c$ and $g : b \to c$ such that for all c' and all $f' : a \to c'$, $g : b \to c'$ there is a unique $k : c \to c'$ such that $f' = k \circ f$, $g' = k \circ g$.

(a) Show that coproducts exist in the category whose objects are all the sets, and whose morphisms are all the functions between sets.

Let A and B be sets and $C = A \times \{0\} \cup B \times \{1\}$ (often called the "disjoint union" of A and B). Let $f : a \mapsto (a, 0)$ for all $a \in A$ and $g : b \mapsto (b, 1)$ for all $b \in B$.

Now if $f' : A \to C'$ and $g' :\to C'$ are functions then clearly there is a unique function k from C to C' that works, namely the function k such that $k : (a, 0) \mapsto f'(a)$ and $k : (b, 1) \mapsto g'(a)$.

(b) Do coproducts exist in the category whose objects are all the abelian groups, and whose morphisms are all the group HMs between abelian groups?

Yes. Let A and B be abelian groups, let $C = A \oplus B$ be the direct sum (this is what commutative algebraists call the product group in this setting). Let $f : A \to C$ be the HM $f : a \mapsto (a, 0)$ and $g : B \to C$ be the HM $f : b \mapsto (0, b)$. Suppose that $f' : A \to C'$ and $g' : B \to C'$ are HMs. If there is k such that $k \circ f = f'$ and $k \circ g = g'$ then it must be the case that

$$k(a, b) = k(a, 0) + k(0, b) = f'(a) + g'(b),$$

and it is routine to check that this formula does define a suitable HM.

(4) (Ex 1.10 from A and M) Let A be a ring and let $\mathcal{R}$ be its nilradical. Show that the following are equivalent:
   (a) A has exactly one prime ideal.
   (b) Every element of A is either a unit or nilpotent.
   (c) $A/\mathcal{R}$ is a field.

   1 implies 2: Since A has exactly one prime ideal, *a fortiori* it is local so the unique prime ideal consists exactly of the non-units. This ideal is also equal to the nilradical so it also consists exactly of the nilpotent elements.

   2 implies 3: Everything not in $\mathcal{R}$ is a unit, so every nonzero coset in $A/\mathcal{R}$ is invertible. (Subtle point of language: the "or" in 2 must be exclusive otherwise the zero ring has property 2).

   3 implies 1: Let P be prime. Then $1 \notin P$ and $\mathcal{R} \subseteq P$ so P corresponds to an ideal of $A/\mathcal{R}$ properly contained in $A/\mathcal{R}$. The only such is the zero ideal so $P = \mathcal{R}$.