COMMUTATIVE ALGEBRA HW 3

JC

Due in class Fri 9 September.

(1) Read the "Categories and products" handout on the course webpage.

(2) Let R be a ring and define a category $\mathcal{C}$ as follows. The objects are all triples (S, a, ϕ) where S is a ring, $\phi : R \to S$ is a ring HM and $a \in S$. The morphisms from (S_1, a_1, ϕ_1) to (S_2, a_2, ϕ_2) are all ring HMs $\psi : S_1 \to S_2$ such that $\psi \circ \phi_1 = \phi_2$ and $\psi(a_1) = a_2$.

Show that $(R[x], x, \phi)$ is an initial object in this category, where ϕ is the usual inclusion map from R to $R[x]$.

(3) Let $\mathcal{C}$ be a category and let a and b be objects in $\mathcal{C}$. A *coproduct* of a and b is an object c together with maps $f : a \to c$ and $g : b \to c$ such that for all c' and all $f' : a \to c'$, $g : b \to c'$ there is a unique $k : c \to c'$ such that $f' = k \circ f$, $g' = k \circ g$.

(a) Show that coproducts exist in the category whose objects are all the sets, and whose morphisms are all the functions between sets.

(b) Do coproducts exist in the category whose objects are all the abelian groups, and whose morphisms are all the group HMs between abelian groups?

(4) (Ex 1.10 from A and M) Let A be a ring and let $\mathcal{R}$ be its nilradical. Show that the following are equivalent:

(a) A has exactly one prime ideal.

(b) Everey element of A is either a unit or nilpotent.

(c) $A/\mathcal{R}$ is a field.