COMMUTATIVE ALGEBRA HW 22 SOLNS

JC

(1) Let R be a graded ring. Say that an ideal I of R is homogeneous iff for every f, $f \in I$ iff every homogeneous component of f is in I.

Show that the following are equivalent

(a) I is a homogeneous ideal.

(b) I is generated by a set of homogeneous elements.

First suppose that I is homogeneous. Then I is generated by the set of homogeneous a such that for some $f \in I$, a is a homogeneous component of f.

Now let I be generated by some set A of homogeneous elements. Let $f \in I$ and write $f = \sum_i r_i a_i$, where (by breaking up elements of R as finite sums of homogeneous elements) we may as well assume that each r_i is homogeneous. Then if f_n is the R_n-component of f we have that $f_n = \sum_{i \in A} r_i a_i$ where $A = \{i : deg(r_i) + deg(a_i) = n\}$, so in particular $f_n \in I$.

2) Fill in the details of the following analysis of P, a nonzero prime ideal in $\mathbb{Z}[x]$.

Before we start an easy remark: $\mathbb{Z}[x]$ is a Noetherian UFD, so if P is a prime ideal and $P = (g_1, \dots g_n)$ then we may choose for each i an irreducible factor f_i of g_i such that $f_i \in P$. Then easily $P = (f_1, \dots f_n)$.

(a) $P \cap \mathbb{Z}$ is either 0 or $p\mathbb{Z}$ for some prime number p.

Proof: $P \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.

(b) If $P \cap \mathbb{Z} = p\mathbb{Z}$ then either $P = (p)$ or $P = (p, f)$ for some irreducible f in $\mathbb{Z}[x]$.

Since $p \in P$ we have $(p) \subseteq P$ so by general facts P corresponds to a prime ideal of $\mathbb{Z}[x]/(p)$, which is IMic to $\mathbb{Z}/p\mathbb{Z}[x]$. Since $\mathbb{Z}/p\mathbb{Z}[x]$ is a PID the prime ideals are of form (G) where G is zero or irreducible.

Lifting we see that either $P = (p)$ or $P = (p, g)$ for some nonzero g. By the argument $P = (p, f)$ for some irreducible f.

Note: Not every irreducible f will work. Consider for example $f = x^2 + p$. A little thought should show that the relevant irreducibles are the ones whose images mod p are also irreducible.

(c) If $P \cap \mathbb{Z} = 0$ then $P = (g)$ for some irreducible $g \in \mathbb{Z}[x]$. Let $P = (f_1, \dots f_n)$ where the f_i are distinct irreducible. By hypothesis they all have positive degree so are primitive, and hence are non-associate irreducibles in $\mathbb{Q}[x]$. Suppose for a contradiction that $n > 1$ and find rational polynomials a and b such that $af_1 + bf_2 = 1$. Multiplyying by a suitable non zero integer m we find integer polynomials A and B with $Af_1 + Bf_2 = m$, so $m \in P$, contradiction!