COMMUTATIVE ALGEBRA HW 18 SOLNS

JC

(1) Verify that if $\{G_n : n \in \mathbb{N}\}$ is a decreasing sequence of subgroups of the group G then
  (a) The collection τ of those sets O such that for all $a \in O$ there is n such that $a + G_n$ is a subset of O forms a topology on G.
  Easy to see τ contains $\emptyset$ and G and is closed under unions. Let O_i for $1 \leq i \leq n$ be in τ with $O = \bigcap_i O_i$. Let $a \in O$, choose for each i some $n(i)$ such that $a + G_{n(i)} \subseteq O_i$; let $n = \max_i n(i)$ and then since the G_j are decreasing we have $a + G_n \subseteq O$.
  (b) This topology makes G into a topological group
  Claim: each set of form $x + G_n$ is open, for if $y \in x + G_n$ then by standard facts about cosets $y + G_n = x + G_n$.
  Enough to show that $a - b$ is continuous. So suppose that O is open with $a - b \in O$, choose n such that $a - b + G_n \subseteq O$. If $U = a + G_n$ and $V = b + G_n$ then $U - V = a - b + G_n \subseteq O$.

(2) Let G be a group and let $\{G_n\}$ and $\{H_n\}$ be two decreasing sequences of subgroups of G. Show that TFAE
  (a) The topology on G induced by $G_n\}$ equals the topology induced by $\{H_n\}$.
  (b) For every m there is n such that $G_n \subseteq H_m$ and for every m there is n such that $H_n \subseteq G_m$.

  a) implies b): $0 \in G_m$ and G_m is open so $H_n \subseteq G_m$ for some n and vv.

  b) implies a): Let O be open in the G topology and let $a \in O$. Find m with $a + G_m \subseteq O$ and n such that $H_n \subseteq G_m$, then $a + H_n \subseteq O$.

  Show that in this case the set of cauchy sequences, the relation of equivalence between sequences and the completion $\hat{G}$ are the same for the two sequences.

  Routine!

(3) True or false? For $H \leq G$ and $\{G_n\}$ a decreasing sequence of subgroups of G, the following two topologies on G/H always coincide:

The topology induced by the decreasing sequence of subgroups $\{\pi_H[G_n] = (G_n + H)/H\}$.

The quotient topology on G/H induced by the map pi_H and the topology on G arising from $\{G_n\}$.

Call them τ_1 and τ_2. $O \subseteq G/H$ is open for τ_1 iff for every $a \in O$ there is n such that $a + \pi_H[G_n] \subseteq O$ iff for every $b \in \pi_H^{-1}[O]$ there is n such that $\pi_H[b + G_n] \subseteq O$ iff iff for every $b \in \pi_H^{-1}[O]$ there is n such that $b + G_n \subseteq \pi_H^{-1}[O]$ iff O is open for τ_2.

(4) Let G be a group, $\{G_n\}$ a decreasing sequence of subgroups and $\hat{G}$ the corresponding completion.

Consider the following two topologies on $\hat{G}$:

The topology on $\hat{G}$ induced by the decreasing sequence of subgroups $\hat{G}_n$.

The topology obtained by giving each G/G_n the discrete topology, giving $\prod_n G/G_n$ the product topology and then giving the inverse limit (which is a subset of $\prod_n G/G_n$) the subspace topology.

What is the relationship between them?

The easiest way to do this is to translate everything into the inverse limit picture and compare bases for the two topologies.

The first topology above has as basis the cosets of the form $a + \hat{G}_n$. In the inverse limit picture $\hat{G}_n$ is the set of sequences $(a_i) \in \lim\limits_{\leftarrow} G/G_i$ such that $a_i = 0$ for $i \leq n$, and so easily the coset $b + \hat{G}_n$ is the set of $(a_i) \in \lim\limits_{\leftarrow} G/G_i$ such that $a_i = b_i$ for $i \leq n$.

Recall that in general the product topology is defined as follows: we take a faily $\{Y_j : j \in I\}$ of topological spaces, form the cartesian product and then take as basic sets the subsets $\prod_j U_j$ where U_j is open in Y_j and $\{j : U_j \neq Y_j\}$ is finite. We will call this the "standard basis".

Now we claim for any sequence of sets $\{X_i : i \in \mathbb{N}\}$, if we equip each set with the discrete topology and then form the product the resulting space has the following basis: basic open sets have the form $O_{\vec{y}} = \{f \in \prod_i X_i : \forall i \leq n; f(i) = y_i\}$ where $\vec{y} \in \prod_{i \leq n} X_i$.

Proof: Each $\{y_i\}$ is open so that easily $O_{\vec{y}_i}$ is in the standard basis. Conversely let $\prod_i U_i$ be in the standard basis, choose n so large that $U_i = X_i$ for $i > n$ and then observe that $\prod_i U_i = \bigcup\{O_{\vec{y}} : \vec{y} \in \prod_{i \leq n} U_i\}$.

So in particular the product topology on $\prod_n G/G_n$ has as a basis sets of form $\{(a_n) : \forall i \leq n \ a_i = b_i\}$. If we intersect these with $\lim\limits_{\leftarrow} G/G_n$ we get a basis whose nonempty elements are precisely of the form $b + \hat{G}_n$. So the two topologies coincide.