COMMUTATIVE ALGEBRA HW 18 SOLNS

JC

(1) Let G be a topological group, N a normal subgroup of G, and $\pi_N : G \to G/N$ the usual projection map $g \mapsto gN = Ng$.

(a) Show that if U is open in G then $UN = NU$, UN is open and $\pi_N[U] = \pi_N[UN]$.
$UN = NU$ because $gN = Ng$ for all $g \in U$. $NU = \bigcup_{g \in N} gU$, the map $h \mapsto gh$ is a topological AM, so gU is open for each $g \in N$ and hence NU is open.
It follows easily that $\pi_N[U]$ is open in G/N, since $\pi_N^{-1}[\pi_N[U]] = UN$.

(b) Give G/N the quotient topology, that is X is open in G iff $\pi_N^{-1}[X]$ is open in G. Show that G/N is a topological group.
We cut a corner by showing that $(g, h) \mapsto gh^{-1}$ is continuous in G/N, this is easily seen to be equivalent to saying that multiplication and inversion are continuous.
So let W be open in G/N with $aN(bN)^{-1} \in W$. We know that $W^* = \pi_N^{-1}[W]$ is open in G, and of course W^* is a union of cosets of N so $W^* = NW^* = W^*N$.
Now $aN(bN)^{-1} = ab^{-1}N = \pi_N(ab^{-1}) \in W$ so that $ab^{-1} \in W^*$. So we may choose $U \ni a$ and $V \ni b$ open in G such that $UV^{-1} \subseteq W^*$. Then by the last remark $\pi[U] \ni aN$ and $\pi[V] \ni bN$ are open in G/N, and now $\pi[U]\pi[V]^{-1} \subseteq \pi[W^*] = W$ as required.

(c) Show that G/N is T_2 iff N is closed in G.
$N = \pi_N^{-1}[\{e\}]$, so N is closed in G iff $\{e\}$ is closed in G/N iff all singletons are closed in G/N iff G is T_1 iff G is T_2. (Using arguments from class after first step!)

(2) (A and M 9.6) Let R be a Dedekind domain and I a nonzero ideal of R. Show every ideal in R/I is principal. Conclude that every ideal of R is generated by at most two elements.

This was a little tricky. Thanks to George Schaeffer for some remarks which simplified my original rather barqoque solution (which was heavy on localisation and Noetherian induction).

We make a series of claims with proofs:

(a) Claim: Let S and T be rings in which every ideal is principal. Then $S \times T$ is also such a ring.
Proof: Suppose K is an ideal of $S \times T$ and let $(a, b) \in K$, then multiplying by $(1, 0)$ and $(0, 1)$ we see that $(a, 0)$ and $(0, b)$ are in K. Since K is also closed under $+$ we see that $(a, b) \in K$ iff both $(a, 0)$ and $(0, b)$ are in K. Clearly $I = \{a : (a, 0) \in K\}$ is an ideal of S so by hypothesis $I = (c)_S$ for some c. Similarly $J = \{b : (0, b) \in K\} = (d)_T$. So now $(a, b) \in K$ iff $(a, 0) \in I$ and $(0, b) \in J$ iff $a \in (c)$ and $b \in (d)$ iff $(a, b) \in ((c, d))$.

(b) Claim: It will suffice to prove the claim when $I = P^n$ for P a nonzero prime and $n \geq 1$.
Proof: Since R is a DD, I is a product $P_1^{e_1} \dots P_n^{e_n}$ for P_i nonzero and prime. The ideals $P_j^{e_j}$ are comaximal in pairs because their radicals P_j are distinct maximal ideals, so that by the CRT $R/I \simeq \prod_j R/P_j^{e_j}$. Now use the previous claim.

(c) Claim: the only ideals of R/P^n are those corresponding to P^i for $0 \leq i \leq n$ (we abuse notation by writing as P^i/P^n for these ideals).
Proof: let $J \supseteq P^n$ be an ideal of R and let Q be any prime ideal appearing in the factorisation of J. Then $P^n \subseteq J \subseteq Q$ and so taking radicals $P \subseteq Q$, hence since P and Q are maximal $P = Q$. So J must be a power of P.

(d) Claim: Each ideal P^i/P^n is principal.
Proof: WLOG we may suppose that $0 < i < n$. By unique factorisation the ideals P_i are distinct, in particular $P^{i+1} \subsetneq P^i$ and so we may choose $a \in P^i \setminus P^{i+1}$.
Now we ask what ideal does $a + P^n$ genarate in R/P^n. It is actually $((a) + P^n)/P^n$ and by the choice of a we see that $(a) + P^n \subseteq P^i$, $(a) + P^n \not\subseteq P^{i+1}$ so that $(a) + P^n = P^i$, so that $a + P^n$ generates P^i/P^n.

(3) Consider a commutative diagram

$$\begin{tikzcd}
 & M_1 \arrow[r] \arrow[d, "\gamma"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r, "\delta"] & N_3 &
\end{tikzcd}$$

where the top line is exact, δ is surjective and β is an isomorphism.

Show that

(a) α is surjective.
This is easy. Choose for any $z \in N_3$ some $y \in N_2$ with $\delta y = z$, then some $x \in M_2$ with $\beta x = y$. If w is the image of x in M_3 then by commutativity $\alpha w = z$.

(b) If γ is surjective and the bottom line is exact, then α is an isomorphism.
We show $ker(\alpha) = 0$. Let $z \in ker(\alpha)$, and choose $y \in M_2$ such that $y \mapsto z$, which is possible by exactness of the top row. By commutativity $\beta y \in ker(\delta)$, so by exactness of the bottom row we may choose $x \in N_1$ so that $x \mapsto \beta y$. Then as γ is surjective we may choose $w \in M_1$ with $\gamma w = x$. Since β is an IM it follows from commutativity that $w \mapsto y$, so by exactness of the top row that $y \mapsto 0 = z$.

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "\gamma"] & y \arrow[r] \arrow[d, "\beta"] & z \arrow[d, "\alpha"] \\
x \arrow[r] & \beta y \arrow[r, "\delta"] & 0
\end{tikzcd}$$

Note: This may look like a weird problem. In fact it is a weird problem. It will save us ten minutes in the middle of an argument about completions next week.