## COMMUTATIVE ALGEBRA HW 18 SOLNS

## JC

- (1) Let G be a topological group, N a normal subgroup of G, and  $\pi_N: G \to G/N$  the usual projection map  $g \mapsto gN = Ng$ .
  - (a) Show that if U is open in G then UN = NU, UN is open and  $\pi_N[U] = \pi_N[UN]$ .

UN = NU because gN = Ng for all  $g \in U$ .  $NU = \bigcup_{g \in N} gU$ , the map  $h \mapsto gh$  is a topological AM, so gU is open for each  $g \in N$  and hence NU is open.

It follows easily that  $\pi_N[U]$  is open in G/N, since  $\pi_N^{-1}[\pi_N[U]] = UN$ .

(b) Give G/N the quotient topology, that is X is open in G iff  $\pi_N^{-1}[X]$  is open in G. Show that G/N is a topological group.

We cut a corner by showing that  $(g, h) \mapsto gh^{-1}$  is continuous in G/N, this is easily seen to be equivalent to saying that multiplication and inversion are continuous.

So let W be open in G/N with  $aN(bN)^{-1} \in W$ . We know that  $W^* = \pi_N^{-1}[W]$  is open in G, and of course  $W^*$  is a union of cosets of N so  $W^* = NW^* = W^*N$ .

Now  $aN(bN)^{-1} = ab^{-1}N = \pi_N(ab^{-1}) \in W$  so that  $ab^{-1} \in W^*$ . So we may choose  $U \ni a$  and  $V \ni b$  open in G such that  $UV^{-1} \subseteq W^*$ . Then by the last remark  $\pi[U] \ni aN$  and  $\pi[V] \ni bN$  are open in G/N, and now  $\pi[U]\pi[V]^{-1} \subseteq \pi[W^*] = W$  as required.

- (c) Show that G/N is  $T_2$  iff N is closed in G.  $N = \pi_N^{-1}[\{e\}]$ , so N is closed in G iff  $\{e\}$  is closed in G/Niff all singletons are closed in G/N iff G is  $T_1$  iff G is  $T_2$ . (Using arguments from class after first step!)
- (2) (A and M 9.6) Let R be a Dedekind domain and I a nonzero ideal of R. Show every ideal in R/I is principal. Conclude that every ideal of R is generated by at most two elements.

This was a little tricky. Thanks to George Schaeffer for some remarks which simplified my original rather barqoque solution (which was heavy on localisation and Noetherian induction).

We make a series of claims with proofs:

(a) Claim: Let S and T be rings in which every ideal is principal. Then  $S \times T$  is also such a ring.

Proof: Suppose K is an ideal of  $S \times T$  and let  $(a, b) \in K$ , then multiplying by (1, 0) and (0, 1) we see that (a, 0) and (0, b) are in K. Since K is also closed under + we see that  $(a, b) \in K$  iff both (a, 0) and (0, b) are in K. Clearly  $I = \{a : (a, 0) \in K\}$  is an ideal of S so by hypothesis  $I = (c)_S$  for some c. Similarly  $J = \{b : (0, b) \in K\} = (d)_T$ . So now  $(a, b) \in K$  iff  $(a, 0) \in I$  and  $(0, b) \in J$  iff  $a \in (c)$ and  $b \in (d)$  iff  $(a, b) \in ((c, d))$ .

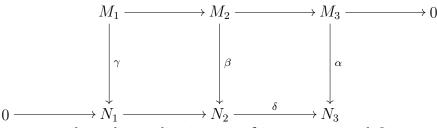
- (b) Claim: It will suffice to prove the claim when  $I = P^n$  for P a nonzero prime and  $n \ge 1$ . Proof: Since R is a DD, I is a product  $P_1^{e_1} \dots P_n^{e_n}$  for  $P_i$ nonzero and prime. The ideals  $P_j^{e_j}$  are comaximal in pairs because their radicals  $P_j$  are distinct maximal ideals, so that by the CRT  $R/I \simeq \prod_j R/P_j^{e_j}$ . Now use the previous claim.
- (c) Claim: the only ideals of  $R/P^n$  are those corresponding to  $P^i$  for  $0 \le i \le n$  (we abuse notation by writing as  $P^i/P^n$  for these ideals).

Proof: let  $J \supseteq P^n$  be an ideal of R and let Q be any prime ideal appearing in the factorisation of J. Then  $P^n \subseteq J \subseteq Q$  and so taking radicals  $P \subseteq Q$ , hence since P and Q are maximal P = Q. So J must be a power of P.

(d) Claim: Each ideal  $P^i/P^n$  is principal. Proof: WLOG we may suppose that 0 < i < n. By unique factorisation the ideals  $P_i$  are distinct, in particular  $P^{i+1} \subsetneq P^i$  and so we may choose  $a \in P^i \setminus P^{i+1}$ .

Now we ask what ideal does  $a + P^n$  genarate in  $R/P^n$ . It is actually  $((a) + P^n)/P^n$  and by the choice of a we see that  $(a) + P^n \subseteq P^i$ ,  $(a) + P^n \notin P^{i+1}$  so that  $(a) + P^n = P^i$ , so that  $a + P^n$  generates  $P^i/P^n$ .

(3) Consider a commutative diagram



where the top line is exact,  $\delta$  is surjective and  $\beta$  is an isomorphism.

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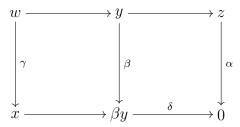
Show that

(a)  $\alpha$  is surjective.

This is easy. Choose for any  $z \in N_3$  some  $y \in N_2$  with  $\delta y = z$ , then some  $x \in M_2$  with  $\beta x = y$ . If w is the image of x in  $M_3$  then by commutativity  $\alpha w = z$ .

(b) If  $\gamma$  is surjective and the bottom line is exact, then  $\alpha$  is an isomorphism.

We show  $ker(\alpha) = 0$ . Let  $z \in ker(\alpha)$ , and choose  $y \in M_2$ such that  $y \mapsto z$ , which is possible by exactness of the top row. By commutativity  $\beta y \in ker(\delta)$ , so by exactness of the bottom row we may choose  $x \in N_1$  so that  $x \mapsto \beta y$ . Then as  $\gamma$  is surjective we may choose  $w \in M_1$  with  $\gamma w = x$ . Since  $\beta$  is an IM it follows from commutativity that  $w \mapsto y$ , so by exactness of the top row that  $y \mapsto 0 = z$ .



Note: This may look like a weird problem. In fact it is a weird problem. It will save us ten minutes in the middle of an argument about completions next week.