COMMUTATIVE ALGEBRA HW 15 SOLUTIONS

JC

Due in class Wed 19 October.

(1) A topological space is called *discrete* if all subsets are open. Show that if R is Artinian then $Spec(R)$ is discrete.

We know that if R is Artinian then $Spec(R)$ is finite and also all points in it are maximal ideals. We saw in a previous HW that P is maximal iff $\{P\}$ is closed, so in our case all points are closed. The set of closed sets is closed under finite unions and $Spec(R)$ is finite so all sets are closed. Hence all sets are open and $Spec(R)$ is discrete.

(2) (A and M 6.4) Let M be a Noetherian R-module and let I be the *annihilator* of M, that is $\{a : \forall m \; am = 0\}$.

Show that R/I is a Noetherian ring.

(Thanks to George Schaeffer for this slick proof).

Let $m_1, \ldots m_n$ generate M. We know that M^n is a Noetherian R-module (easy induction using the fact that for any R, if $M \leq N$ then N is Noetherian iff both M and N/M are Noetherian). Consider the R-module HM from R to M^n given by $r \mapsto (rm_1, \ldots rm_n)$. The kernel is I so by the first IM thm R/I is IMic as an R-module to a submodule of M^n. So R/I is a Noetherian R-module. A little thought shows that this is equivalent to R/I being a Noetherian ring.

Show by example that R need not be Noetherian.

Let R be any non-Noetherain ring and $M = 0$.

(3) Let R be a local Noetherian integral domain of dimension one.

Show that R has a unique nonzero prime ideal P, and that every ideal $I \neq 0, R$ of R is P-primary.

Since R is an ID $\{0\}$ is prime. For P any nonzero prime $0 \subsetneq P$ must be a maximal chain of prime ideals, hence P is maximal, and so is equal to the unique maximal ideal.

Let $I \neq 0, R$. Then I is a finite intersection of nonzero primary ideals, which must all be P-primary as P is the only nonzero prime ideal, and hence I is P-primary.