COMMUTATIVE ALGEBRA HW 14

JC

Due in class Wed 12 October.

(1) Recall that

   (a) When $E \leq F$ are fields and $\alpha \in F$ is algebraic over E, the *minimal polynomial* of α over E is by definition the unique monic $m \in E[x]$ such that $(m)_{E[x]} = \{f \in E[x] : f(\alpha) = 0\}$.

   (b) A complex number β is an *algebraic integer* iff β is integral over $\mathbb{Z}$.

   Let $\beta \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that β is an algebraic integer iff the minimal polynomial of β over $\mathbb{Q}$ has integer coefficients. Hint: use Gauss' lemma for the trickier direction.

   If the minimal poly has integer coefficients then it witnesses that β is an algebraic integer. Conversely let β be an integer and fix a monic $g \in \mathbb{Z}[x]$ with $g(\beta) = 0$. Let m be the minimal poly of β over $\mathbb{Q}$. Then m divides g in $\mathbb{Q}[x]$ say $g = mq$. Now let M and Q be the unique primitive associates of m and q with positive leading coefficients; then MQ is a primitive associate of g with positive leading coefficient that is $g = MQ$, but then M and Q must be monic so $m = M \in \mathbb{Z}[x]$.

(2) (A and M 6.1.i) Let M be a Noetherian R-module and $\phi : M \to M$ an R-module HM. By considering the modules $Ker(\phi^n)$ show that if ϕ is surjective then ϕ is an R-module IM.

   The modules $Ker(\phi^n)$ form an increasing chain which must stabilise. Let $Ker(\phi^n) = Ker(\phi^{n+1})$. We claim that $Ker(\phi) = 0$. For if $\phi(y) = 0$ then (since ϕ^n is obviously surjective) we can write $y = \phi^n(x)$, then $\phi(y) = \phi^{n+1}(x) = 0$ so $x \in ker(\phi^n)$ and hence $y = \phi^n(x) = 0$.

(3) (A and M 5.5) Let A and B be rings with $A \leq B$ and B integral over A. Show that

   (a) If $a \in A$ is a unit in B then it is a unit in A.

   (b) The Jacobson radical of A is the contraction of the Jacobson radical of B.

Let $a \in A$ be a unit in B, then a^{-1} is integral over A say $a^{-n} = \sum_{i<n} a_i a^{-i}$ for $a_i \in A$. Multiply by a^{n-1} to conclude that $a^{-1} \in A$.

We could probably do the second part this way but we prefer to use the powerful theorems on integral extensions and prime ideals.

Since B is integral over A we know that for every maximal ideal M of B, $M \cap A$ is maximal in A. Why? Well M maximal in B implies M prime in B implies $M \cap A$ prime in A, and now we use the argument from class. Conversely if $\bar{M}$ is maximal in A then $\bar{M}$ is prime in A so by another result from class there is M prime in B with $M \cap B = A$, but then M is maximal in B since $\bar{M}$ is maximal in A.

Since the Jacobson radical is the intersection of maximal ideals we are done.