# COMMUTATIVE ALGEBRA HW 14

JC

Due in class Wed 12 October.

(1) Recall that
   (a) When $E \leq F$ are fields and $\alpha \in F$ is algebraic over $E$, the *minimal polynomial* of $\alpha$ over $E$ is by definition the unique monic $m \in E[x]$ such that $(m)_{E[x]} = \{f \in E[x] : f(\alpha) = 0\}$.
   (b) A complex number $\beta$ is an *algebraic integer* iff $\beta$ is integral over $\mathbb{Z}$.

   Let $\beta \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that $\beta$ is an algebraic integer iff the minimal polynomial of $\beta$ over $\mathbb{Q}$ has integer coefficients. Hint: use Gauss' lemma for the trickier direction.

(2) (A and M 6.1.i) Let $M$ be a Noetherian $R$-module and $\phi : M \to M$ an $R$-module HM. By considering the modules $Ker(\phi^n)$ show that if $\phi$ is surjective then $\phi$ is an $R$-module IM.

(3) (A and M 5.5) Let $A$ and $B$ be rings with $A \leq B$ and $B$ integral over $A$. Show that
   (a) If $a \in A$ is a unit in $B$ then it is a unit in $A$.
   (b) The Jacobson radical of $A$ is the contraction of the Jacobson radical of $B$.