COMMUTATIVE ALGEBRA HW 14

JC

Due in class Wed 12 October.

(1) Recall that
  (a) When $E \leq F$ are fields and $\alpha \in F$ is algebraic over E, the *minimal polynomial* of α over E is by definition the unique monic $m \in E[x]$ such that $(m)_{E[x]} = \{f \in E[x] : f(\alpha) = 0\}$.
  (b) A complex number β is an *algebraic integer* iff β is integral over $\mathbb{Z}$.

  Let $\beta \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that β is an algebraic integer iff the minimal polynomial of β over $\mathbb{Q}$ has integer coefficients. Hint: use Gauss' lemma for the trickier direction.

(2) (A and M 6.1.i) Let M be a Noetherian R-module and $\phi : M \to M$ an R-module HM. By considering the modules $Ker(\phi^n)$ show that if ϕ is surjective then ϕ is an R-module IM.

(3) (A and M 5.5) Let A and B be rings with $A \leq B$ and B integral over A. Show that
  (a) If $a \in A$ is a unit in B then it is a unit in A.
  (b) The Jacobson radical of A is the contraction of the Jacobson radical of B.