COMMUTATIVE ALGEBRA HW 1

JC

Due in class Fri 2 September. Uses the ideas in the proof of the theorem "R UFD implies $R[x]$ UFD".

(1) Let R be a PID, let $X \subseteq R$ be a set with at least one nonzero element and (X) the ideal generated by X. Show that $a \in R$ is a gcd for X iff $(a) = (X)$.

(2) Let R be a UFD, let F be the field of fractions of R. Show that
   (a) If f and g are primitive and irreducible in $R[x]$ and f divides g in $F[x]$, then f and g are associates in $R[x]$.
   (b) If f and g are non-associate irreducibles in $R[x]$ then
      (i) f and g have no common irreducible factor in $F[x]$.
      (ii) There exist $A, B \in R[x]$ and $c \in R$ such that $Af + Bg = c$, $c \neq 0$.
      Hint: what can you say about the ideal generated by f and g in the PID $F[x]$?

(3) Let k be a field and let $A = k[x, y]$ be the ring of polynomials in two variables x and y with coefficients from k.
   (a) Prove that A is a UFD (Hint: $A \simeq k[x][y]$).
   (b) Let $f, g \in A$ be non-associate irreducible polynomials. Show that there are only finitely many $(c, d) \in k^2$ such that $f(c, d) = g(c, d) = 0$.
   Hint: let $R = k[x]$, let $F = k(x)$ (the field of fractions of R) and appeal to the preceding question.
   Hint: what do you know about the number of roots of a nonzero polynomial in $k[x]$?