COMMUTATIVE ALGEBRA HANDOUT: MORE TOPOLOGY AND TOPOLOGICAL GROUPS

JC

1. More topology

1.1. **Bases and subbases.** A *basis* for a topology τ is a family $\mathcal{B}$ of open sets of τ such that any open set of τ is a union of sets in $\mathcal{B}$. It should be clear that if X is a set and $\mathcal{B}$ is a family of subsets of X such that the union of $\mathcal{B}$ is X and the intersection of any two members of $\mathcal{B}$ is a union of members of $\mathcal{B}$, then $\mathcal{B}$ is a basis for a unique topology τ on X; the open sets of τ are precisely the sets which are unions of sets in $\mathcal{B}$. This gives a convenient way of defining a topology.

Example: In a metric space the open balls form a basis.

Example: In $Spec(R)$ the sets $O_a = \{P : a \notin P\}$ form a basis.

More generally given any family $\mathcal{S}$ of subsets we may look at the least topology containing $\mathcal{S}$. Exercise: which sets are in this topology? If τ is the least topology containing $\mathcal{S}$ we say that $\mathcal{S}$ is a *subbasis* for τ; note that any basis for τ is automatically a subbasis for τ.

1.2. **More on continuity.** By definition if we are given topological spaces X and Y, then a function $f : X \to Y$ is *continuous* iff for every open $V \subseteq Y$, $f^{-1}[V]$ is open in X. Here are some alternative formulations (the proofs that these properties are equivalent to continuity are left as an exercise). We assume that we have fixed a basis $\mathcal{B}$ for the topology on X and a basis $\mathcal{C}$ for the topology pn Y.

(1) f is continuous iff for every closed $F \subseteq Y$, $f^{-1}[F]$ is closed in X.
(2) f is continuous iff for every $V \in \mathcal{C}$, $f^{-1}[V]$ is open in X.
(3) f is continuous iff for every $x \in X$ and every open set $V \subseteq Y$ with $f(x) \in V$, there is an open set $U \subseteq X$ such that $x \in U$ and $f[U] \subseteq V$.
(4) f is continuous iff for every $x \in X$ and every $V \in \mathcal{C}$ with $f(x) \in V$, there is $U \in \mathcal{B}$ such that $x \in U$ and $f[U] \subseteq V$.

1.3. **Separation axioms.** Let X be a topological space. We consider various axioms saying that points can be separated by open sets.

Axiom T_0: For all $a, b \in X$ with $a \neq b$ EITHER there is an open set U such that $a \in U$ and $b \notin U$ OR there is an open set V such that $b \in V$ and $a \notin V$.

Axiom T_1; For all $a, b \in X$ with $a \neq b$ there is an open set U such that $a \in U$ and $b \notin U$.

Axiom T_2: For all $a, b \in X$ with $a \neq b$ there are open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

Remarks:

(1) T_2 spaces are usually called *Hausdorff spaces.*
(2) The T_1 axiom says that for every point b, $\{b\}$ is closed.

In most areas of mathematics the spaces that arise are T_2, for example any metric space is T_2. However $Spec(R)$ is only T_0 in general; if P and Q are prime ideals with $P \subsetneq Q$ then we can choose $a \in Q \setminus P$ so that $P \in O_a$ and $Q \notin O_a$, but every open set which contains Q also contains P.

1.4. **Closure and interior.** Let X be a topological space and $Y \subseteq X$. Then the union of all the open sets contained in Y is open and is called the *interior* of Y, written $int(Y)$. Dually the intersection of all the closed sets containing Y is closed and is called the *closure* of Y, written $cl(Y)$.

If $A \subseteq X$ we write A^c for $X \setminus A$. Naive set theory gives that $int(A)^c = cl(A^c)$ and dually $cl(A)^c = int(A^c)$.

Y is *dense* in X iff the closure of Y is X. Since $cl(Y) = int(Y^c)^c$ this amounts to saying that $int(Y^c)$ is empty, or to put it another way every nonempty open set meets Y.

1.5. **Product and quotient topology.** Let X and Y be topological spaces. We define the product topology on $X \times Y$ as the topology with basis the sets $U \times V$ with U open in X and V open in Y. So a set is open in the product topology iff it is a union of such "open rectangles".

Remarks

(1) If we give $\mathbb{R}$ and $\mathbb{R}^2$ the usual metrics, then the topology we get on $\mathbb{R}^2$ really is obtained by forming the product of two copies of the topology on $\mathbb{R}$. To put it another way every open ball is a union of open rectangles.
(2) Suppose that τ is any topology on $X \times Y$ which makes the projection maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$ continuous. Then $U \times Y$ and $X \times V$ are in τ for every open U in X and open V in Y, hence $(U \times Y) \cap (X \times V) = U \times V$ is in τ. Put more concisely: the product topology is the smallest topology which makes the projectiosn continuous.
(3) The product topology on $X \times Y$ with the projection maps really is a product in the category of topological spaces and continuous maps.

Let X be a topological space, Y a set and $\pi : X \to Y$ a function. Then the *quotient topology* on Y is the set of $A \subseteq Y$ such that $\pi^{-1}[A]$ is open in X. Routinely this is a topology, and is in fact the largest topology on Y which makes π continuous.

1.6. **Compactness.** Let X be a topological space and Y a subset of X. An *open covering* of Y is a collection $\mathcal{O}$ of open subsets of X such that $Y \subseteq \bigcup \mathcal{O}$. If $\mathcal{O}$ is an open covering of Y then a *subcovering* of $\mathcal{O}$ is a subset of $\mathcal{O}$ which is also an open covering of Y. The set Y is *compact* iff every open covering of Y has a finite subcovering.

Remark: We say that a space X is *compact* iff X is a compact subset of X. It is easy to see that Y is a compact subset of X iff Y becomes a compact space when equipped with the subspace topology.

Example: The famous Heine-Borel theorem says that $[0, 1]$ is a compact subset of $\mathbb{R}$.

Remark: Let $\mathcal{B}$ be a basis for the topology on X. Then it is easy to see that X is a compact space iff every open covering $\mathcal{O} \subseteq \mathcal{B}$ has a finite subcovering.

Remark: Older books (including A and M) sometimes reserve the word "compact" for spaces which are T_2 and use the word "quasicompact" for the property that every open cover has a finite subcover.

Remark: It is easy to see that a closed subset of a compact space is compact.

Naive set theory gives us an equivalent version of compactness in terms of closed sets. We say that a family $\mathcal{F}$ of subsets of X has the *finite interesction property (fip)* iff for every finite $f \subseteq \mathcal{F}$, the intersection of f is nonempty. Compactness can be reformulated thus: for every family $\mathcal{F}$ of closed sets, if $\mathcal{F}$ has the finite interesction property then the inetresction of $\mathcal{F}$ nonempty.

1.7. **Spectra and varieties as topological spaces.** We show that $Spec(R)$ is compact. As we pointed out above it's enough to show that every covering by basic open sets has a finite subcover. So suppose that $Spec(R)$ is the union of O_a for $a \in A$. This means that for every prime ideal P there is $a \in A$ with $a \notin P$. So if I is the ideal generated by A then we must have $I = R$ (for otherwise $I \subseteq P$ with P prime and hence $a \in P$ for all $a \in A$). So $1 \in I$, so $1 \in (a_1, \dots a_k)_R = R$ for some finite set of $a_i \in A$. But now arguing as above we have $Spec(R) = \bigcup_i O_{a_i}$.

Similarly if k is an algebraically closed field and $V \subseteq \mathbb{A}^n(k)$ is a variety then V is compact in the Zariski topology. We leave this as an exercise (Hints: if $I = I(V)$ then by the Nullstellensatz the points of V correspond to the maximal ideals containing I. Use this to verify the version of compactness that involves families of closed sets with the fip.)

1.8. **Irreducible sets and Noetherian topological spaces.** (This is just an axiomatic version of something we did in the special case of varieties in an earlier handout)

Recall that a closed set is *irreducible* iff it is nonempty and is not the union of two proper closed subsets. Recall also that a closed subset Y of X is irreducible iff it is an irreducible space when equipped with the subspace topology.

Definition: a space is *Noetherian* iff every decreasing sequence of closed subsets stabilises (equivalently iff every nonempty family of closed sets has a minimal element).

In the algebraic geometry handout we actually proved (think about it) that any closed set in a Noetherian space can be written uniquely as an irredundant finite union of irreducible closed sets.

The following facts are easy:

(1) A nonempty space X is irreducible iff every nonempty open set is dense, that is to say every two nonempty open sets intersect.
(2) Any nonempty open subset of an irreducible space is itself irreducible in the subspace topology.