COMMUTATIVE ALGEBRA HANDOUT 2: CATEGORIES AND PRODUCTS

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It is common in mathematics to consider a class of structures (say the class of all groups or the class of all metric spaces) together with a class of "structure preserving maps" between the structures (say HMs in the case of groups or continuous maps in the case of metric spaces). The notion of "category" is obtained by abstracting the essential features of this setup.

We will use category theory as a language in which to describe and explain some phenomena in algebra, and also as a labour-saving device. Of course we are only brushing the surface of this area of mathematics here; for more see the courses on Category Theory and Categorical Logic or Maclane's classic "Categories for the working mathematician".

Definition 1. *A* category $\mathcal{C}$ *consists of*

1. *A class $Ob(\mathcal{C})$ of "objects".*
2. *A class $Hom(\mathcal{C})$ of "morphisms" together with two functions dom ("domain") and cod ("codomain") from $Hom(\mathcal{C})$ to $Ob(\mathcal{C})$. If $dom(f) = a$ and $cod(f) = b$ we write $f : a \to b$, and we let $Hom(a, b)$ be the class of all $f : a \to b$.*

   IMPORTANT: The morphisms need not be functions and even if they are then dom and cod do not necessarily have their standard meanings. Hence the scare quotes! See examples below.
3. *A "composition operation" $\circ$ which assigns to each pair (g, f) from $Mor(\mathcal{C})$ with $cod(f) = dom(g)$ an element $g \circ f$ of $Mor(\mathcal{C})$ such that $dom(g \circ f) = dom(f)$, $cod(g \circ f) = cod(g)$.*

These must satisfy two simple axioms: $\circ$ is associative (that is $h \circ (g \circ f) = (h \circ g) \circ f$ whenever $cod(f) = dom(g)$ and $cod(g) = dom(h)$) and there exists for each a an element $id_a \in Hom(a, a)$ (an "identity element") such that $f \circ id_a = f$ for all b and all $f : a \to b$, and $id_a \circ g = g$ for all b and all $g : b \to a$.

We note that id_a is unique, because if id'_a has the same properties then $id_a = id_a \circ id'_a = id'_a$.

Here are some examples:

- **Sets**: the class of objects is the class of all sets, and $Hom(a, b)$ is the set of all functions with domain a and codomain b. Composition and identity have the standard meanings.
- **Groups**: the class of objects is the class of all groups, and $Hom(a, b)$ is the set all HMs from a to b. (We could make lots of categories in a similar way using other sorts of algebraic structures) Composition and identity have the standard meanings.

- Recall that a poset (partially ordered set) is a set P equipped with a binary relation $\leq$ which is transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$), reflexive ($a \leq a$) and antisymmetric ($a \leq b \leq a$ implies $a = b$).

  We can make a poset $(P, \leq)$ into a category as follows: the objects are the elements of P, $Hom(a, b)$ contains a unique morphism when $a \leq b$ and is empty otherwise. Composition and identity are defined in the only possible way.

  Any category where $|Hom(a, b)| \leq 1$ for all a and b arises in this way.
- Given a category $\mathcal{C}$ we can obtain an "opposite category" $\mathcal{C}^{op}$ by "reversing all the arrows". Formally $\mathcal{C}^{op}$ is the category $\mathcal{D}$ such that $Ob(\mathcal{C}) = Ob(\mathcal{D})$, $Mor(\mathcal{C}) = Mor(\mathcal{D})$, $dom_{\mathcal{C}} = cod_{\mathcal{D}}$, $dom_{\mathcal{D}} = cod_{\mathcal{C}}$. $g \circ_{\mathcal{D}} f = f \circ_{\mathcal{C}} g$.

Remark: In a poset category the morphisms are not functions. In $\mathbf{Sets}^{op}$ the morphisms are functions but the *dom* and *cod* operators are not the usual ones.

If a and b are objects in a category we say they are *isomorphic* iff there exist $f : a \to b$ and $g : b \to a$ such that $f \circ g = id_b$ and $g \circ f = id_a$. As usual f is an *isomorphism (IM)* iff it has a two-sided inverse as above; in general isomorphic objects may be isomorphic in may ways, for example in **Sets** any two sets of the same cardinality are isomorphic and there are generally many bijections. It is especially interesting when isomorphic objects are isomorphic in exactly one way.

An object a in a category $\mathcal{C}$ is *initial* iff for every object b there exists a unique morphism $f : a \to b$. Examples: the empty set in **Sets** or any group of order one in **Groups**. We see less trivial examples soon. Initial objects amy not exist but if they do they are unique in the following strong sense.

Theorem 1. *If a and b are both initial there is a unique IM $f : a \to b$.*

Proof. Let $f : a \to b$ and $g : b \to a$ be the unique morphisms from a to b and from b to a. Then $g \circ f : a \to a$ and also $id_a : a \to a$, but since a is initial there is exactly one morphism from a to a. Hence $g \circ f = id_a$ and similarly $f \circ g = id_b$. $\square$

This simple proof can be used to prove many uniqueness theorems (and the main trick will always be to cook up a suitable category). Here are some examples.

A *terminal object* is an object b such that for every a there is a unique morphism $f : a \to b$. We claim any two such are isomorphic via a unique IM. To see this just observe that the terminal objects in $\mathcal{C}$ are exactly the initial objects of $\mathcal{C}^{op}$.

If a and b are objects then a *product* of a and b is an object c together with morphisms $f : c \to a$ and $g : c \to b$ with the following "universal property"; for every object d and pair of morphisms $f' : d \to a$ and $g' : d \to b$ there is a unique morphism $k : d \to c$ so that $f' = f \circ k$ and $g' = g \circ k$. Example: if G and H are groups then $G \times H$ with the HMs $(g, h) \mapsto g$ and $(g, h) \mapsto h$ constitute a product in the category of groups.

We claim that products are unique in the following strong sense: if c_1, f_1, g_1 and c_2, f_2, g_2 are both products of a and b there is a unique IM $k : c_1 \to c_2$ such that $f_1 = f_2 \circ k$, $g_1 = g_2 \circ k$. To see this we define a new category whose objects are triples $(c, f : c \to a, g : c \to b)$, in which a morphism from (c, f, g) to (c', f', g') is $h : c \to c'$ such that $f = f' \circ h$ and $g = g' \circ h$; now a product of a and b in the original category is precisely a terminal object of this new category, and the claim is immediate by uniqueness of terminal objects.

Remark: It is important that the product is not just an object but an object together with some maps.

Remark: Suppose P is a poset and a, b are elements of P. Then by definition a product is some $c \leq a, b$ such that for all d if $d \leq a, b$ then $d \leq c$. That is a product is just a greatest lower bound.