COMMUTATIVE ALGEBRA HANDOUT 4: AFFINE ALGEBRAIC GEOMETRY OVER AN ALGEBRAICALLY CLOSED FIELD

JC

Let k be an algebraically closed field. We define the *affine n-space* $\mathbb{A}^n(k)$ to be the set of all n-tuples from k.

Note: You may wonder why we do not use the more familiar notation k^n in this context. The answer is that k^n suggests the usual n-D VS whose group of symmetries is all invertible linear maps; in the current context we want to think of the symmetries of $\mathbb{A}^n(k)$ as being all the invertible *affine maps*. Recall that if V is a VS an affine map from V to V is a map of form $v \mapsto Tv + w$ where $T : V \to V$ is linear and $w \in V$.

Let $A = k[x_1, \ldots x_n]$, then for any set $P \subseteq A$ we define

$$V(P) = \{\vec{a} \in \mathbb{A}^n(k) : \forall f \in P \; f(\vec{a}) = 0$$

Clearly if $I = (P)_A$ then $V(I) = V(P)$. Sets of the form $V(I)$ are called (affine) *varieties*.

Notational warning: some writers (notably Hartshorne) reserve the word "variety" for the objects that we will call "irreducible varieties", and dub the sets of the form $V(I)$ the "algebraic sets".

The ring A is Noetherian so we may find a finite set $B \subseteq I$ such that $I = (B)_A$. Then $V(I) = V(B)$, and we showed that every variety is the set of common zeroes of a finite set of polynomials.

For $X \subseteq \mathbb{A}^n(k)$ we define

$$I(X) = \{f \in A : \forall \vec{a} \in X \; f(\vec{a}) = 0\}$$

Clearly $I(X)$ is an ideal.

The following facts are immediate: for all $X, Y \subseteq \mathbb{A}^n$ and $P, Q \subseteq A$

(1) $X \subseteq Y \to I(Y) \subseteq I(X)$, $P \subseteq Q \to V(Q) \subseteq V(P)$.
(2) $X \subseteq V(I(X))$, $P \subseteq I(V(P))$.
(3) $I(X) = I(V(I(X)))$, $V(P) = V(I(V(P)))$.

We are really interested in using the V and I operators to set up a correspondence between ideals and varieties. From the equations above if V is a variety then $V = V(I(V))$. However not every ideal is the ideal of a variety, in fact we have from the Nullstellensatz that for an ideal J

$$I(V(J)) = \sqrt{J}.$$

Now notice that $\sqrt{\sqrt{J}} = \sqrt{J}$. It follows that the ideals of form $I(V)$ are precisely the *radical ideals*, that is to say the ideals J such that $J = \sqrt{J}$. Note that

$$V(J) = V(I(V(J))) = V(\sqrt{J}),$$

so we might as well just have used the radical ideals to define varieties. The maps I and V set up an inclusion-reversing bijection between the varieties and the radical ideals of A.

Recall that if $\{I_\alpha\}$ is a family of ideals in some ring R then $\sum_\alpha I_\alpha$ is the ideal of all finite sums $\sum_{i=1}^n a_i$ where $a_i \in I_{\alpha_i}$ for some α_i. This is the least ideal containing each I_α. It is routine to check that if the I_α are ideals in A then $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$. A similarly trivial argument shows that $I(\bigcup_\alpha X_\alpha) = \bigcap_\alpha I(X_\alpha)$.

As for unions of varieties recall that if I and J are ideals of a ring R then IJ is the ideal generated by all products ab with $a \in I$, $b \in J$. Clearly $IJ \subseteq I \cap J \subseteq I$ so $V(I) \subseteq V(I \cap J) \subseteq V(IJ)$. Hence $V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(IJ)$. On the other hand if $\vec{a} \notin V(I) \cup V(J)$ we may choose $f \in I$ and $g \in J$ with $f(\vec{a}), g(\vec{a}) \neq 0$ and then $fg(\vec{a}) \neq 0$ so that $\vec{a} \notin V(IJ)$. In conclusion $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.

Zariski topology: We just saw the the class of varieties is closed under arbitrary intersections and finite unions. It follows (why?) that for any variety V the set of varieties W such that $W \subseteq V$ can be seen as the set of closed sets of a certain topology on V, the *Zariski topology.*

Note: For any topological space X and any subset $Y \subseteq X$ we may form the *subspace* topology on Y as follows: the open subsets of Y are all sets of form $O \cap Y$ where O is open in X, or equivalently the closed subsets of Y are all sets of form $F \cap Y$ for F closed in X. In general a subset of Y which is closed in the subspace topology on Y need not be closed in the topology on X, but if Y is closed in the topology on X then a subset of Y is closed in the topology on Y iff it is closed in the topology on X.

Easy exercise: Convince yourself the Zariski topology on V can be obtained by starting with the Zariski topology on $\mathbb{A}^n(k)$ and then inducing the subset topology on V as above.

Note: Recall that a topological space is *irreducible* iff it is nonempty and is not the union of two properly smaller closed sets. We say that a closed set in a topological set is *irreducible* iff it is nonempty and is not the union of two properly smaller closed sets. It is not hard to see (check it!) that if Y is a closed subset of X then Y is an irreducible subset of X iff it is an irreducible space in the subspace topology.

Coordinate ring: Each polynomial $f \in A$ induces a function ($\bar{f}$ say) from $\mathbb{A}^n(k)$ to k. Let V be a variety. Then $f \mapsto \bar{f} \restriction V$ is a surjective ring HM from A to the ring of "polynomial functions" from V to k. The kernel is exactly $I(V)$ so by the first IM theorem the ring of polynomial functions on V is IMIc to the quotient ring $A/I(V)$. We call the ring $A/I(V)$ the *coordinate ring* of the variety V and denote it by $A(V)$.

Cultural note: Actually the usual map $k \hookrightarrow A$ makes A into a k-algebra, the map which takes each element in k to the corresponding constant function on V is also a HM which makes the ring of polynomial functions on V a k-algebra, and $f \mapsto \bar{f} \restriction V$ is a HM of k-algebras in the natural sense. So $A/I(V)$ and the ring of polynomial functions on V are isomorphic as k-algebras.

Irreducible varieties: since prime ideals are radical it is natural to ask which varieties answer to the prime ideals. We will prove that V is irreducible iff $I(V)$ is prime.

Note: by the discussion above V is an irreducible subset of $\mathbb{A}^n(k)$ iff it is ireducible when considered as a topological space in its own right!

Suppose first that I is prime and let $V = V(I)$ (so $I = I(V)$). We know $V(I) \neq \emptyset$ by the Nullstellensatz. Suppose that $V(I) = V(J) \cup V(K)$ for some radical J and K. Then $I = I(V(I)) = IV(J) \cap IV(K) = J \cap K$, so $I \subseteq J$ and $I \subseteq K$. We claim that $I = J$ or $I = K$; for otherwise we may choose $a \in J \setminus I$ and $b \in K \setminus I$ and then $ab \in J \cap K \setminus I$.

Now suppose that I is radical and not prime. If $I = A$ then $V(I) = \emptyset$. Otherwise there are $a, b \notin I$ such that $ab \in I$. Let $J = \sqrt{I + (a)}$ and $K = \sqrt{I + (b)}$ so that J and K are radical. Now $I \subsetneq J$, so by the bijection between radical ideals and varieties $V(J) \subsetneq V(I)$ and similarly $V(K) \subsetneq V(I)$. We claim that $JK \subseteq I$; to see this let $x \in J$ and $y \in K$, choose n so large that $x^n \in I + (a)$ and $y^n \in I + (b)$, then $(xy)^n \in I + (ab) = I$ so that $xy \in \sqrt{I} = I$. But now $V(I) \subseteq V(JK) = V(J) \cup V(K)$ so that $V(I) = V(J) \cup V(K)$ and fails to be irreducible.

Cultural note: If V is irreducible then the coordinate ring $A(V) = A/I(V)$ is an ID so it has a field of fractions. We may think of the elements of this field as quotients f/g of polynomial functions on V where g is not identically zero on V; it is quite possible that g vanishes at some points of V so that f/g may not have a well-defined value at all points of V.

Decomposition of a variety: We will show that any variety is a finite union of irreducible varieties in an essentially unique way. To be a bit more precise we look at *irredundant* unions in which none of the irreducible varieties is a proper subset of another one.

Existence: The key point is that (by the correspondence between varieties and ideals and the Noetherian-ness of polynomial rings) every decreasing sequence of varieties must stabilise; so just as in the case of ideals we may argue that every nonempty set of varieties has a minimal element. Let X be the set of varieties which are not finite unions of irreducible varieties, then if X is nonempty we may choose a minimal $W \in X$. W is not irreducible so $W = W_1 \cup W_2$ where the W_i are varieties properly contained in W, then by the minimality of W they are finite unions of irreducible varieties. Contradiction! To obtain an irredundant union just take any finite union which uses the minimal possible number of irreducible varieties.

Uniqueness: Let $V = W_1 \cup \dots W_m = W'_1 \cup \dots W'_n$ be irredundant unions of irreducible varieties. Now for each i we have $W_i = (W_i \cap W'_1) \cup \dots (W_i \cap W'_n)$ so as W_i is irreducible we must have $W_i = W_i \cap W'_j$ for some j, that is $W_i \subseteq W'_j$. Symetrically there exists k such that $W'_j \subseteq W_k$, so that $W_i \subseteq W'_j \subseteq W_k$. The union of the W_i is irredundant so $W_i = W_k$ and hence $W_i = W'_j$. Now easily $m = n$ and $\{W_1, \dots W_m\} = \{W'_1, \dots W'_n\}$.